Math 222A – Homework 5 (Due Oct 23)

1. Consider the static Hamilton-Jacobi equation

$$\text{(H)} \quad \begin{cases} u_{x_1}u_{x_2} = 1 & \text{in } U \\ u = 0 & \text{on } \Gamma := \partial U, \end{cases}$$

where

$$U = \left\{ x \in (0,\infty)^2 \ : \ x_1 > 1 \ \text{ or } \ x_2 > 1 \right\}.$$

Let $A = (0,1)$, $B = (1,1)$ and $C = (1,0)$.

(a) Show that there are no admissible triples (p^0, z^0, x^0) with $x^0 \in \Gamma \setminus \{A, B, C\}$.

(b) For each $x^0 \in \{A, B, C\}$, find all admissible triples (p^0, z^0, x^0). At nonsmooth boundary points $x^0 \in \{A, B, C\}$, we call a triple (p^0, z^0, x^0) admissible if

$$p_1^0 p_2^0 = 1 \text{ and } z^0 = u(x^0).$$

(c) For each $x^0 \in \{A, B, C\}$ and each admissible triple from part (b), solve the characteristic equations for (H).

(d) For each $x \in U$, find all possible values of $u(x)$ by tracing characteristics back from x to Γ. This gives a "multi-valued solution" of (H). Which points $x \in U$ lie on exactly one characteristic curve? Which lie on exactly two characteristic curves? Which lie three?

(e) Choose a single value for $u(x)$ at each x so that u is continuous on $\overline{U}$ and differentiable almost everywhere in U. Write down your solution in closed form and check that it solves (H) at all points of differentiability. Sketch the characteristics for your solution.

2. Solve using method of characteristics

(a) $x_1 u_{x_1} + x_2 u_{x_2} = 2u, \quad u(x_1, 1) = g(x_1).$

(b) $u u_{x_1} + u_{x_2} = 1, \quad u(x_1, x_1) = \frac{1}{2}x_1.$

3. Let $u, v \in C^1(U_T) \cap C(\overline{U_T})$ satisfy

$$\begin{cases} u_t + H(Du, x) \le 0 & \text{in } U_T \\ u \le g & \text{on } \Gamma_T, \end{cases}$$

and

$$\begin{cases} v_t + H(Dv, x) \ge 0 & \text{in } U_T \\ v \ge g & \text{on } \Gamma_T. \end{cases}$$

Show that $u \le v$ on $\overline{U_T}$. Conclude that there is at most one solution $u \in C^1(U_T) \cap C(\overline{U_T})$ of the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(Du, x) = 0 & \text{in } U_T \\ u = g & \text{on } \Gamma_T, \end{cases}$$

[Hint: Let $\varepsilon > 0$ and set $w(x,t) = u(x,t) - v(x,t) - \varepsilon t$. Show that w attains its maximum on the boundary Γ_T.]

4. Evans: Section 3.5, Problem 10 (Problem 5 in 1st edition)
5. Evans: Section 3.5, Problem 11 (Problem 6 in 1st edition)
6. Evans: Section 3.5, Problem 13 (Problem 7 in 1st edition)
7. Evans: Section 3.5, Problem 14 (Problem 8 in 1st edition)
8. Evans: Section 3.5, Problem 19 (Problem 13 in 1st edition)
9. Evans: Section 3.5, Problem 20 (Problem 14 in 1st edition)