MATH 222A – HOMEWORK 6 (DUE NOV 4)

1. Show that the condition $v \in C_c^\infty(\mathbb{R} \times [0,\infty))$ can be replaced by $v \in C_c^1(\mathbb{R} \times [0,\infty))$ in the definition of integral solution of scalar conservation laws.

2. Let $g \in L^\infty(\mathbb{R})$ be compactly supported, $T > 0$, and let u be the entropy solution of the scalar conservation law

$$\text{(S)} \quad \begin{cases} u_t + F(u)_x = 0 & \text{in } \mathbb{R} \times (0,T] \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

where F is smooth and uniformly convex. Let $\varepsilon > 0$, set $M = \varepsilon^{-1}$, and suppose $u^\varepsilon \in C^\infty([-M,M] \times [0,T])$ is a solution of the viscous conservation law

$$(\mathrm{S}_\varepsilon) \quad \begin{cases} u_t^\varepsilon + F(u^\varepsilon)_x - \varepsilon u_{xx}^\varepsilon = 0 & \text{in } (-M,M) \times (0,T] \\ u_x^\varepsilon = 0 & \text{on } \{\pm M\} \times (0,T] \\ u^\varepsilon = g^\varepsilon & \text{on } [-M,M] \times \{t = 0\}, \end{cases}$$

where $g^\varepsilon = \eta^\varepsilon * g$, and η^ε is the standard mollifier.

   (a) Show that
   $$\|u^\varepsilon\|_{L^\infty([-M,M]\times[0,T])} \leq \|g\|_{L^\infty(\mathbb{R})}. \tag{1}$$
   [Hint: Use a maximum principle argument.]

   (b) Since F is uniformly convex, there exists $\theta > 0$ such that $F'' \geq \theta$. Show that
   $$u_x^\varepsilon(x,t) \leq \frac{1}{\theta t} \quad \text{for all } (x,t) \in [-M,M] \times (0,T]. \tag{2}$$
   Explain, briefly, why you *cannot* similarly bound the absolute value $|u_x^\varepsilon|$. [Hint: Set $a(x,t) = u_x^\varepsilon(x,t)$ and differentiate (S_ε) to find a PDE satisfied by a. Show that $v(x,t) = \frac{1}{\theta t}$ is a supersolution of this PDE and use a comparison principle argument similar to HW #5, Problem 3 to establish (2).]

   (c) Suppose that the limit $\lim_{\varepsilon\to 0^+} u^\varepsilon(x,t)$ exists for almost every $(x,t) \in \mathbb{R} \times [0,T]$. Show that $u^\varepsilon \to u$ pointwise almost everywhere as $\varepsilon \to 0$. [Actually, we need not assume the limit exists. It is an interesting exercise to use the Banach-Alaoglu Theorem to show that u^ε converges to u in the weak* topology on L^∞.]

3. Evans: Section 3.5, Problem 17 (Problem 11 in 1st edition)

4. Evans: Section 4.7, Problem 1

5. Evans: Section 4.7, Problem 2

6. Find a nonnegative scaling invariant solution having the form
$$u(x,t) = t^{-\alpha} v(xt^{-\beta})$$
of the nonlinear heat equation
$$u_t - \Delta(u^\gamma) = 0$$
where $\frac{n-2}{n} < \gamma < 1$. Your solution should go to zero algebraically as $|x| \to \infty$.

7. Find a solution of

$$-\Delta u + u^{\frac{n+2}{n-2}} = 0 \quad \text{in } B(0,1)$$

having the form $u = \alpha(1 - |x|^2)^{-\beta}$ for positive constants α and β. This example shows that a solution of a nonlinear PDE can be finite within a region and yet approach infinity everywhere on its boundary.