Math 5467 – Homework 3

Instructions:

- Complete the problems below, and submit your solutions by uploading them to your shared Google form for HW3.
- If you use LaTeX to write up your solutions, upload them as a pdf file. Students who use LaTeX to write up their solutions will receive bonus points on the homework assignment.
- If you choose to write your solutions and scan them, please either use a real scanner, or use a smartphone app that allows scanning with you smartphone camera. It is not acceptable to submit images of your solutions, as these can be hard to read.

Problems:

1. (Split-Radix FFT) Assume $n \geq 4$ is a power of 2 and let $f \in L^2(\mathbb{Z}_n)$. Define $f_e \in L^2(\mathbb{Z}_{\frac{n}{2}})$, and $f_{o,1}, f_{o,2} \in L^2(\mathbb{Z}_{\frac{n}{4}})$ by

$$f_e(k) = f(2k), \quad f_{o,1}(k) = f(4k+1), \quad \text{and} \quad f_{o,2}(k) = f(4k+3).$$

   (i) Show that

$$\mathcal{D}_n f(\ell) = \mathcal{D}_{\frac{n}{2}} f_e(\ell) + e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell). \tag{1}$$

   (ii) The FFT algorithm based on the 3-way split in (1) is called the split-radix FFT algorithm. There are a lot of redundant computations in (1), and these must be accounted for in order to realize the improved complexity of the split-radix FFT. Show that

$$\begin{aligned}
\mathcal{D}_n f(\ell) &= \mathcal{D}_{\frac{n}{2}} f_e(\ell) + \ (e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)),\\
\mathcal{D}_n f(\ell + \tfrac{n}{2}) &= \mathcal{D}_{\frac{n}{2}} f_e(\ell) - \ (e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) + e^{-2\pi i3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)),\\
\mathcal{D}_n f(\ell + \tfrac{n}{4}) &= \mathcal{D}_{\frac{n}{2}} f_e(\ell + \tfrac{n}{4}) - i(e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) - e^{-2\pi i3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)),\\
\mathcal{D}_n f(\ell + \tfrac{3n}{4}) &= \mathcal{D}_{\frac{n}{2}} f_e(\ell + \tfrac{n}{4}) + i(e^{-2\pi i\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,1}(\ell) - e^{-2\pi i3\ell/n}\mathcal{D}_{\frac{n}{4}} f_{o,2}(\ell)),
\end{aligned}$$

   for $0 \leq \ell \leq \frac{n}{4} - 1$. This gives all the outputs of $\mathcal{D}f(\ell)$ and reduces the number of multiplications and additions required.

   (iii) Explain how the observations in Part (ii) allow you to compute $\mathcal{D}_n f$ from $\mathcal{D}_{\frac{n}{2}} f_e$, $\mathcal{D}_{\frac{n}{4}} f_{o,1}$ and $\mathcal{D}_{\frac{n}{4}} f_{o,2}$ using $6n$ real operations. [Note, multiplications with ± 1 or $\pm i$ do not count, since they amount to negation of real or imaginary parts, which can be absorbed into the next operation by changing it from addition to subtraction or vice versa]

   (iv) Show that part (iii) implies that the number of real operations taken by the split-radix FFT, denoted again as A_n, satisfies the recursion

$$A_n = A_{\frac{n}{2}} + 2A_{\frac{n}{4}} + 6n.$$

Explain why $A_1 = 0$ and $A_2 = 4$. Use this to show that $A_n \leq 4n \log_2 n$. [Hint: Define $B_n = A_n - 4n \log_2 n$ and show that B_n satisfies

$$B_n = B_{\frac{n}{2}} + 2B_{\frac{n}{4}}$$

with $B_1 = 0$ and $B_2 = -4$. Use this to argue that $B_n \leq 0$ for all power-of-two n.] [Note: If one is more careful about redundant computations (there are additional multiplications with ± 1 or $\pm i$ that can be skipped), then the complexity of the split-radix FFT algorithm is actually $4n \log_2 n - 6n + 8$ real operations].

2. Discrete derivatives (difference quotients) can be interpreted as convolutions. Complete the following exercises.

   (i) For $f \in L^2(\mathbb{Z}_n)$ define the backward difference

   $$\nabla^- f(k) = f(k) - f(k-1).$$

   Find $g \in L^2(\mathbb{Z}_n)$ so that $\nabla^- f = g * f$ and use the DFT convolution property $\mathcal{D}(g * f) = \mathcal{D}g\mathcal{D}f$ to show that $\mathcal{D}(\nabla^- f)(k) = (1 - \omega^{-k})\mathcal{D}f(k)$, where $\omega = e^{2\pi i/n}$.

   (ii) For $f \in L^2(\mathbb{Z}_n)$ define the forward difference

   $$\nabla^+ f(k) = f(k+1) - f(k).$$

   Find $g \in L^2(\mathbb{Z}_n)$ so that $\nabla^+ f = g * f$ use this to show that $\mathcal{D}(\nabla^+ f)(k) = (\omega^k - 1)\mathcal{D}f(k)$.

   (iii) For $f \in L^2(\mathbb{Z}_n)$ define the centered difference by

   $$\nabla f(k) = \frac{1}{2}(\nabla^- f(k) + \nabla^+ f(k)) = \frac{1}{2}(f(k+1) - f(k-1)).$$

   Use parts (i) and (ii) to show that

   $$\mathcal{D}(\nabla f)(k) = \frac{1}{2}(\omega^k - \omega^{-k})\mathcal{D}f(k) = i \sin(2\pi k/n)\mathcal{D}f(k).$$

   (iv) For $f \in L^2(\mathbb{Z}_n)$, define the discrete Laplacian as

   $$\Delta f(k) = \nabla^+ \nabla^- f(k) = f(k+1) - 2f(k) + f(k-1).$$

   Use parts (i) and (ii) to show that

   $$\mathcal{D}(\Delta f)(k) = (\omega^k + \omega^{-k} - 2)\mathcal{D}f(k) = 2(\cos(2\pi k/n) - 1)\mathcal{D}f(k).$$

3. Consider the Poisson equation

   $$\Delta u = f \quad \text{on } \mathbb{Z}_n. \tag{2}$$

   The source term $f \in L^2(\mathbb{Z}_n)$ is given, and $u \in L^2(\mathbb{Z}_n)$ is the unknown we wish to solve for. The discrete Laplacian Δ is defined in Problem 2. Use the DFT and the results from Problem 2 to derive a solution formula for u using one forward transform $\mathcal{D}$ and one inverse transform $\mathcal{D}^{-1}$. Is there a condition you need to place on $\mathcal{D}f$ for your solution formula to make sense? [Hint: Take the DFT of both sides of (2), solve for $\mathcal{D}u$, and then apply the inverse DFT $\mathcal{D}^{-1}$. Be careful not to divide by zero when you solve for $\mathcal{D}u$.]

4. Let $n \geq 1$ be odd. Show that for $t \notin \mathbb{Z}$ we have

$$\frac{1}{n} \sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} e^{2\pi i k t} = \frac{\operatorname{sinc}(nt)}{\operatorname{sinc}(t)}.$$

What happens when $t \in \mathbb{Z}$? Here, sinc is the normalized sinc function $\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$.