Mathematics of Image and Data Analysis
Math 5467

Parseval's Identities and Convolution

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Last time

- The Fast Fourier Transform (FFT)

Today

- Parseval's identities
- Convolution and the DFT

Recall

Definition 1. The *Discrete Fourier Transform (DFT)* is the mapping $\mathcal{D} : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$\mathcal{D}f(\ell) = \sum_{k=0}^{n-1} f(k)\omega^{-k\ell} = \sum_{k=0}^{n-1} f(k)e^{-2\pi i k\ell/n},$$

where $\omega = e^{2\pi i/n}$ and $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ is the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n$.

The DFT can be viewed as a change of basis into the orthogonal basis functions

$$u_\ell(k) = \omega^{k\ell} = e^{2\pi i k\ell/n} = \cos(\cdots) + i\sin(\cdots)$$

for $\ell = 0, 1, \ldots, n-1$.

Inverse Fourier Transform

Theorem 2 (Fourier Inversion Theorem)**.** *For any* $f \in L^2(\mathbb{Z}_n)$ *we have*

$$f(k) = \frac{1}{n}\sum_{\ell=0}^{n-1} \mathcal{D}f(\ell)\omega^{k\ell} = \frac{1}{n}\sum_{\ell=0}^{n-1} \mathcal{D}f(\ell)e^{2\pi i k\ell/n}. \tag{1}$$

Definition 3 (Inverse Discrete Fourier Transform)**.** The *Inverse Discrete Fourier Transform (IDFT)* is the mapping $\mathcal{D}^{-1} : L^2(\mathbb{Z}_n) \to L^2(\mathbb{Z}_n)$ defined by

$$\mathcal{D}^{-1}f(\ell) = \frac{1}{n}\sum_{k=0}^{n-1} f(k)\omega^{k\ell} = \frac{1}{n}\sum_{k=0}^{n-1} f(k)e^{2\pi i k\ell/n}.$$

$$\langle f, g\rangle = \sum_{k=0}^{n-1} f(k)\overline{g(k)}$$

Adjoint of $\mathcal{D}$

We first show that $\mathcal{D}^{-1}$ is the adjoint of $\mathcal{D}$, up to the factor $1/n$.

$$\mathcal{D}^{-1} f(\ell) = \frac{1}{n}\sum_{k=0}^{n-1} f(k)\omega^{k\ell}$$

Lemma 4. *For each $f, g \in L^2(\mathbb{Z}_n)$ we have*

$$\frac{1}{n}\langle \mathcal{D}f, g\rangle = \langle f, \mathcal{D}^{-1}g\rangle.$$

$\underline{\text{Proof}}$: $\langle f, \mathcal{D}^{-1}g\rangle = \sum_{k=0}^{n-1} f(k)\overline{\mathcal{D}^{-1}g(k)}$

$$= \sum_{k=0}^{n-1} f(k)\,\overline{\frac{1}{n}\sum_{\ell=0}^{n-1} g(\ell)\omega^{\ell k}}$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} f(k)\sum_{\ell=0}^{n-1}\overline{g(\ell)}\,\omega^{-\ell k}$$

$$= \frac{1}{n}\sum_{\ell=0}^{n-1}\overline{g(\ell)}\sum_{k=0}^{n-1} f(k)\,\omega^{-\ell k}$$

$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

$\omega = e^{2\pi i/n}$

$\overline{\omega} = \omega^{-1}$

$$\overline{\omega^{\ell k}} = \omega^{-\ell k}$$

$Df(\ell)$

$$\mathcal{D}f(\ell) = \sum_{k=0}^{n-1} f(k)\,\omega^{-k\ell}$$

$$= \frac{1}{n} \sum_{\ell=0}^{n-1} \overline{g(\ell)}\, Df(\ell)$$

$$= \frac{1}{n} \langle Df, g \rangle .$$

∎

Parseval's identities

An immediate consequence of the adjoint lemma is Parseval's identities.

Theorem 5 (Parseval's Identities)**.** *Let $f, g \in L^2(\mathbb{Z}_n)$. Then it holds that*

(i) $\langle f, g\rangle = \frac{1}{n}\langle \mathcal{D}f, \mathcal{D}g\rangle$, *and*

(ii) $\|f\|^2 = \frac{1}{n}\|\mathcal{D}f\|^2$.

$$\frac{1}{n}\langle \mathcal{D}f, g\rangle = \langle f, \mathcal{D}^{-1}g\rangle. \quad (*)$$

Proof: (i) $\langle f, g\rangle = \langle f, D^{-1}Dg\rangle \overset{(*)}{=} \frac{1}{n}\langle Df, Dg\rangle.$

(ii) $\|f\|^2 = \langle f, f\rangle \overset{(i)}{=} \frac{1}{n}\langle Df, Df\rangle$

$= \frac{1}{n}\|Df\|^2.$

Recall: $x, y \in \mathbb{R}^n$, $x \cdot y = \cos\theta \, \|x\| \, \|y\|$

(i) says the "angles" between vectors are preserved/unchanged by the DFT

(ii) lengths of vectors are unchanged by the DFT

(i)+(ii) DFT is a rotation!

Aside: Mathematically nicer to use definitions

$$Df(\ell) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(k)\,\omega^{-k\ell}$$

$$D^{-1}f(\ell) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(k)\,\omega^{+k\ell}$$

$$DD^{-1} = I = D^{-1}D$$

The Parseval is $\langle f, g\rangle = \langle Df, Dg\rangle$

$\|f\| = \|Df\|$.

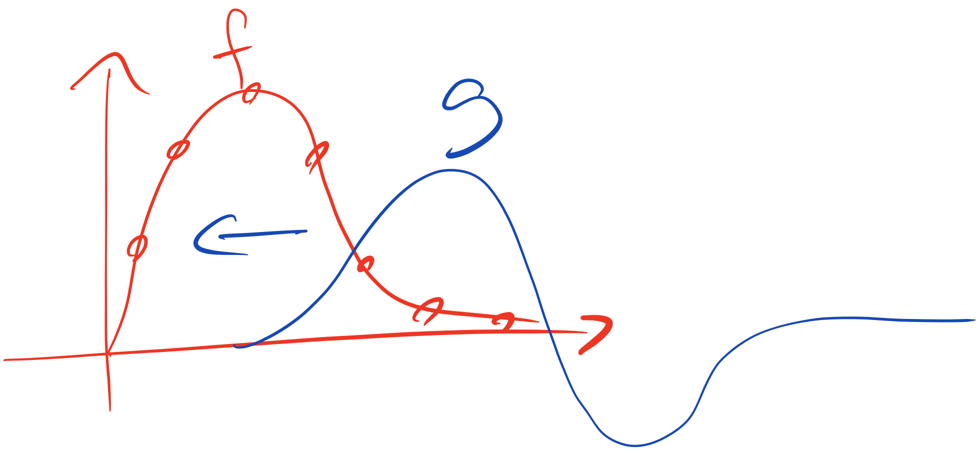
Adjoint of $D = D^* = D^{-1}$

Parseval's identities

Remark 6. Of course, a similar statement holds for the inverse transform $\mathcal{D}^{-1}$. Indeed, Lemma 4 and Theorem 2 imply

$$\frac{1}{n}\langle f, g\rangle = \frac{1}{n}\langle \mathcal{D}\mathcal{D}^{-1} f, g\rangle = \langle \mathcal{D}^{-1} f, \mathcal{D}^{-1} g\rangle.$$

Setting $f = g$ yields $\frac{1}{n}\|f\|^2 = \|\mathcal{D}^{-1} f\|^2$.



Convolution

Definition 7. The *discrete cyclic convolution* of $f, g \in L^2(\mathbb{Z}_n)$, denoted $f * g$, is the function in $L^2(\mathbb{Z}_n)$ defined for each k by

$$\boxed{(f * g)(k) = \sum_{j=0}^{n-1} f(j)g(k-j).} \quad g(j-k)$$

We note that the definition of the convolution makes use of the fact that $\mathbb{Z}_n$ is a cyclic group when $k - j$ falls outside of $0, 1, \ldots, n-1$ (i.e., the values wrap around). We leave some basic properties of the convolution to an exercise.

Exercise 8. Let $f, g, h \in L^2(\mathbb{Z}_n)$. Show that the following hold.

(i) $f * g = g * f$;

(ii) $f * (g * h) = (f * g) * h$;

(iii) $f * (g + h) = f * g + f * h$.

HW #3

△

Convolution and the DFT

Lemma 9 (Convolution and the DFT). *For $f, g \in L^2(\mathbb{Z}_n)$ we have*

$$\boxed{\mathcal{D}(f * g) = \mathcal{D}f \cdot \mathcal{D}g.} \tag{2}$$

Remark 10. Lemma 9 is the most important property of the DFT, that it turns convolution into multiplication. It allows us to compute convolutions with the FFT in $O(n \log n)$ operations as

$$f * g = \mathcal{D}^{-1}(\mathcal{D}f \cdot \mathcal{D}g).$$

Computing convolution the ordinary way takes $O(n^2)$ operations:

$$(f * g)(k) = \sum_{j=0}^{n-1} f(j) g(k-j). \quad O(n)$$

The convolution property is also what allows the FFT to be used for solving PDEs numerically (all discrete derivatives are convolutions).

Proof: $D(f*g)(\ell) = \sum_{k=0}^{n-1} (f*g)(k)\, w^{-k\ell}$

$$= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f(j)\, g(k-j)\, w^{-k\ell}$$

$$= \sum_{j=0}^{n-1} f(j) \sum_{k=0}^{n-1} g(\underbrace{k-j}_{q})\, w^{-k\ell} \qquad q = k-j, \quad k = q+j, \quad q: -j \longrightarrow n-1-j$$

$$= \sum_{j=0}^{n-1} f(j) \sum_{q=-j}^{n-1-j} g(q)\, w^{-(q+j)\ell}$$

$$(*) = \sum_{j=0}^{n-1} f(j)\, w^{-j\ell} \sum_{q=-j}^{n-1-j} g(q)\, w^{-q\ell}$$

$$= \sum_{q=0}^{n-1} g(q)\, w^{-q\ell}$$

$$= Dg(\ell)$$

$$(*) = \left[\sum_{j=0}^{n-1} f(j)\, w^{-j\ell}\right] Dg(\ell)$$

$$D(f*g)(\ell) = Df(\ell) \cdot Dg(\ell). \qquad \blacksquare$$

$$\sum_{q=-j}^{n-1-j} \underbrace{g(q)\,\omega^{-q\ell}}_{h(q)} = \underbrace{\sum_{q=-j}^{-1} h(q)} + \sum_{q=0}^{n-1-j} h(q)$$

$$q = k-n, \quad k = q+n$$

$$k : n-j \longrightarrow n-1$$

$$= \sum_{k=n-j}^{n-1} h(k-n) + \sum_{q=0}^{n-1-j} h(q)$$

Since h is n-periodic $h(k-n) = h(k)$

$$= \sum_{k=n-j}^{n-1} h(k) + \sum_{k=0}^{n-1-j} h(k)$$

$$= \sum_{k=0}^{n-1} h(k).$$

Exercise on discrete derivatives HW#3

Exercise 11. Discrete derivatives (difference quotients) can be interpreted as convolutions. Complete the following exercises.

(i) For $f \in L^2(\mathbb{Z}_n)$ define the backward difference

$$\nabla^- f(k) = f(k) - f(k-1).$$

Find $g \in L^2(\mathbb{Z}_n)$ so that $\nabla^- f = f * g$ and use this with Lemma 9 to show that $\boxed{\mathcal{D}(\nabla^- f)(k) = (1-\omega^{-k})\mathcal{D}f(k)}$, where $\omega = e^{2\pi i/n}$.

(ii) For $f \in L^2(\mathbb{Z}_n)$ define the forward difference

$$\nabla^+ f(k) = f(k+1) - f(k).$$

Find $g \in L^2(\mathbb{Z}_n)$ so that $\nabla^+ f = f * g$ and use this with Lemma 9 to show that $\mathcal{D}(\nabla^+ f)(k) = (\omega^k - 1)\mathcal{D}f(k)$.

(iii) For $f \in L^2(\mathbb{Z}_n)$ define the centered difference by

$$\nabla f(k) = \frac{1}{2}(\nabla^- f(k) + \nabla^+ f(k)) = \frac{1}{2}(f(k+1) - f(k-1)).$$

Use parts (i) and (ii) to show that

$$\mathcal{D}(\nabla f)(k) = \frac{1}{2}(\omega^k - \omega^{-k})\mathcal{D}f(k) = i\sin(2\pi k/n)\mathcal{D}f(k).$$

(iv) For $f \in L^2(\mathbb{Z}_n)$, define the discrete Laplacian as

$$\Delta f(k) = \nabla^+ f(k) - \nabla^- f(k) = f(k+1) - 2f(k) + f(k-1).$$

Use parts (i) and (ii) to show that

$$\mathcal{D}(\Delta f)(k) = (\omega^k + \omega^{-k} - 2)\mathcal{D}f(k) = \underbrace{2(\cos(2\pi k/n) - 1)}\mathcal{D}f(k). \qquad \triangle$$

(i) $\nabla^- f(k) = f(k) - f(k-1)$

$$(f * g)(k) = \sum_{j=0}^{n-1} f(j)\, g(k-j)$$

$j = k$

$j = k-1 \Rightarrow k-j = 1$

$g(0) = 1$

$g(1) = -1$

$g(k) = 0, \quad k = 2, 3, \dots, n-1$

$$\nabla^- f(k) = (f * g)(k).$$

$$D(\nabla^- f)(k) = Df(k)\, Dg(k)$$

Theorem from class

$$Dg(k) = \sum_{\ell=0}^{n-1} g(\ell)\, w^{-\ell k}$$

$$= g(0)\, w^0 + g(1)\, w^{-k}$$

$$= 1 - \omega^{-k}$$

$$\boxed{D(\nabla^{-} f)(k) = (1 - \omega^{-k})\, Df(k)}$$