MATH 5587 – HOMEWORK 11 (DUE THURSDAY DEC 8)

The terms distribution and generalized function are used interchangeably in the homework. Recall a function f is **locally integrable** if

$$\int_a^b |f(x)|\,dx < \infty \quad \text{for all } a < b.$$

1. (a) Show that $x\delta(x) = 0$.
   (b) Show that $\delta(2x) = \frac{1}{2}\delta(x)$.
   (c) Let $\delta^{(k)}$ denote the k^{th} distributional derivative of the Delta function. Show that

   $$\langle \delta^{(k)}, \varphi \rangle = (-1)^k \varphi^{(k)}(0)$$

   for all test functions φ.

2. Let

   $$f_n(x) = \begin{cases} \frac{n}{2}, & \text{if } -\frac{1}{n} < x < \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$$

   Show that $f_n \rightharpoonup \delta$ weakly as $n \to \infty$. Sketch the graph of the functions f_n.

3. Let f_n be the sequence of generalized functions given by

   $$f_n(x) = \frac{n}{2}\left(\delta\left(x + \tfrac{1}{n}\right) - \delta\left(x - \tfrac{1}{n}\right)\right).$$

   Show that

   $$f_n \rightharpoonup \delta'(x) \quad \text{weakly as } n \to \infty.$$

4. Let $\varphi \in C_c^\infty(\mathbb{R})$ and let f be a generalized function. Show that the product rule

   $$(\varphi f)' = \varphi' f + \varphi f'$$

   holds in the distributional sense.

5. In this question, you will construct a test function that is positive on the interval $(-1, 1)$ and vanishes outside of this interval. Recall a test function must be infinitely differentiable and compactly supported.

   Define

   $$\psi(x) = \begin{cases} 0, & x \le 0 \\ e^{-\frac{1}{x^2}}, & x > 0. \end{cases}.$$

   (a) Use induction to show that for $x > 0$ **all** derivatives of ψ of all orders are of the form

   $$p\left(\tfrac{1}{x}\right) e^{-\frac{1}{x^2}},$$

   where p is a polynomial. [The polynomial p will be different for each derivative. You don't need to find p explicitly.]

(b) For any polynomial p show that

$$\lim_{x\to 0^+} p\left(\tfrac{1}{x}\right) e^{-\frac{1}{x^2}} = 0.$$

[Hint: By setting $y = 1/x$, you may show instead that

$$\lim_{y\to\infty} p(y)e^{-y^2} = 0.$$

To prove this, first show that for any positive integer k, $\lim_{y\to\infty} y^k e^{-y^2} = 0$. There are many ways to do this. One way is to use the Taylor series for e^x to show that $e^x \geq x^k/k!$ for any $x > 0$. From this you can deduce that $e^{-y^2} \leq k!/y^{2k}$.]

(c) Conclude from (b) that ψ is infinitely differentiable.

(d) Define

$$\varphi(x) = \psi(x+1)\psi(1-x).$$

Show that φ is infinitely differentiable, positive in $(-1, 1)$ and vanishes for $x \notin (-1, 1)$. [Hint: The product and composition of smooth functions is smooth.]

(e) Fix $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. Use part (d) to find a test function g that is positive on the interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ and vanishes outside of this interval. [Hint: Scale and translate φ.]

6. (a) Show that $f(x) = \frac{1}{2}\log(x^2)$ is locally integrable, and thus defines a generalized function.

(b) Show that the ordinary derivative $f'(x) = \frac{1}{x}$ is not locally integrable, and is therefore not a generalized function.

(c) Show that the distributional derivative of f is the distribution

$$\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = \text{P.V.} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\, dx.$$

[Here, P.V. stands for the Cauchy principle value of the integral, and is defined as

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\, dx := \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx,$$

where

$$\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\, dx = \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}\, dx.$$

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(d) Show that the second distributional derivative of f is the distribution

$$\langle f'', \varphi\rangle = \langle f, \varphi''\rangle = \text{P.V.} \int_{-\infty}^{\infty} \frac{\varphi(0) - \varphi(x)}{x^2}\, dx.$$