Math 5587 – Homework 12 Solutions

Note: We are using single bars |·| instead of double ∥·∥ for the norm of vectors in solutions below.

1. Find the Green’s function for the tilted half-space \( D = \{(x, y, z) : ax + by + cz > 0\} \).

[Hint: Use the Green’s function for the halfspace \( \{z > 0\} \) and a change of variables.]

*Solution.* The Green’s function will have the form
\[
G(x, x_0) = -\frac{1}{4\pi|x - x_0|} + \frac{1}{4\pi|x - x_0^*|},
\]
where \( x_0^* \) is a reflected point to be determined outside of the domain. Let \( \alpha = \sqrt{a^2 + b^2 + c^2} \) and let \( n = [a/\alpha, b/\alpha, c/\alpha]^T \) be the unit vector in the normal direction to the boundary of \( D \). Hence \( |n| = 1 \),
\[
D = \{x \in \mathbb{R}^3 : n \cdot x > 0\}
\]
and
\[
\partial D = \{x \in \mathbb{R}^3 : n \cdot x = 0\}.
\]
We want to reflect \( x_0 \) through the plane \( x \cdot n = 0 \), hence we define
\[
x_0^* = x_0 - 2(x_0 \cdot n)n,
\]
and define the Green’s function as above.

To check that this is the Green’s function, we need to check that \( G(x, x_0) = 0 \) for \( x \cdot n = 0 \) (i.e., on the boundary), and that \( x_0^* \notin D \), so that the corrector is harmonic in \( D \). Since
\[
x_0^* \cdot n = x_0 \cdot n - 2(x_0 \cdot n)n \cdot n = -x_0 \cdot n < 0,
\]
we have that \( x_0^* \notin D \), and hence \( H(x) = \frac{1}{4\pi|x-x_0^*|} \) is harmonic in \( D \). Finally, for \( x \) on the boundary we have \( x \cdot n = 0 \) and hence
\[
|x-x_0^*|^2 = |x-x_0 + 2(x_0 \cdot n)n|^2 = |x-x_0|^2 + 4(x-x_0) \cdot (x_0 \cdot n)n + 4|\langle x_0 \cdot n \rangle|^2.
\]
Therefore
\[
|x-x_0^*|^2 = |x-x_0|^2 + 4(x_0 \cdot n)x \cdot n - 4(x_0 \cdot n)x_0 \cdot n + 4(x_0 \cdot n)^2.
\]
Since \( x \cdot n = 0 \) we have \( |x-x_0| = |x-x_0^*| \) for \( x \) on the boundary, hence \( G(x, x_0) = 0 \) for \( x \) on the boundary of \( D \). This verifies that \( G \) is the desired Green’s function.

2. Find the Green’s function for the half ball
\[
D = \{(x, y, z) : x^2 + y^2 + z^2 < a^2 \text{ and } z > 0\}.
\]

[Hint: Reflect the solution for the whole ball across the plane \( z = 0 \).]
Solution. Let $G$ be the Green’s function for the whole ball, which has the form
\[
G(x, x_0) = \frac{1}{4\pi|x - x_0|} + \frac{a}{4\pi|x_0||x - x_0^*|},
\]
where $x_0^* = \frac{a^2 x_0}{|x_0|^2}$. We claim that the Green’s function for $D$ is
\[
G(x, x_0) = \overline{G}(x, x_0) - \overline{G}(x, \hat{x}_0),
\]
where $\hat{x}_0 = (x_0, y_0, -z_0)$. Writing this out we have
\[
G(x, x_0) = \frac{1}{4\pi|x - x_0|} + \frac{a}{4\pi|x_0||x - x_0|} + \frac{1}{4\pi|x - \hat{x}_0|} - \frac{a}{4\pi|\hat{x}_0||x - \hat{x}_0^*|},
\]
where $\hat{x}_0^* = (x_0^*, y_0^*, -z_0^*)$. All the points $x_0^*$, $\hat{x}_0$ and $\hat{x}_0^*$ lie outside of the upper half of the unit ball, so the final three terms above are harmonic on $D$.

It remains to see that $G(x, x_0) = 0$ for $x \in \partial D$. The boundary is composed of two pieces. First, there are points $x \in \partial D$ with $|x| = a$ and $z > 0$. Here, $\overline{G}(x, x_0) = \overline{G}(x, \hat{x}_0) = 0$, hence $G(x, x_0) = 0$. Then there is a portion of the boundary where $z = 0$. Here, just like for the Green’s function for the half-space, if $x = (x, y, 0)$ then
\[
|x - x_0| = |x - \hat{x}_0| \quad \text{and} \quad |x - x_0^*| = |x - \hat{x}_0^*|.
\]
Since we clearly also have $|x_0| = |\hat{x}_0|$, $G(x, x_0) = 0$ for $x = (x, y, 0)$. This verifies that $G$ is the Green’s function.

3. Consider the two dimensional disk
\[
D = \{(x, y) : x^2 + y^2 < a^2\}.
\]

Show that the Green’s function for the disk is
\[
G(x, x_0) = \frac{1}{2\pi} \log(||x - x_0||) - \frac{1}{2\pi} \log\left(\frac{||x_0||}{a} ||x - x_0^*||\right),
\]
where $x_0^* = \frac{a^2 x_0}{||x_0||^2}$.

4. Use problem 3 to recover the two dimensional version of Poisson’s formula for the ball that we derived in class using separation of variables.