1. For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous.

(a) \( u_t - u_{xx} + x = 0 \). 2nd order, inhomogeneous linear

(b) \( (x + y)u_{xy} + 2xu_y = x^2 \). 2nd order, inhomogeneous linear

(c) \( u_{xx} = e^u \). 2nd order, nonlinear

(d) \( u_{txy} - u_{xxx}u_{yy} + u_x = x^3 \). 3rd order, nonlinear

(e) \( u_t + u_{xxxxx} - \sqrt{1 + u^2} = 0 \). 6th order, nonlinear

(f) \( u_t + u_x + u_y + u/xy = 0 \). 1st order, homogeneous linear

2. Let \( u_* \) be a solution of the inhomogeneous linear equation \( L[u_*] = g \). Show that every solution of \( L[u] = g \) is of the form \( u = u_* + v \), where \( v \) is a solution of the homogeneous linear equation \( L[v] = 0 \).

Solution. Let \( u \) be a solution of \( L[u] = g \) and set \( v := u - u_* \). Since \( L \) is linear
\[
L[v] = L[u - u_*] = L[u] - L[u_*] = g - g = 0.
\]

3. Find the solution to the initial value problem \( u_t + u_x = 0 \) satisfying \( u(x, 1) = x/(1 + x^2) \).

Solution. The left hand side of the PDE is the directional derivative in the direction \((1, 1)\) in the \((x, t)\) plane. Hence, for fixed \(x_0\) consider the function \( g(t) = u(x_0 + t, t) \). Then
\[
g'(t) = u_t(x_0 + t, t) + u_x(x_0 + t, t) = 0,
\]
and so \( g \) is constant. Therefore
\[
u(x_0 + t, t) = g(t) = g(1) = u(x_0 + 1, 1) = \frac{x_0 + 1}{1 + (x_0 + 1)^2}.
\]
Now set \( x = x_0 + t \), so \( x_0 = x - t \) to find that
\[
u(x, t) = \frac{x - t + 1}{1 + (x - t + 1)^2}.
\]

4. Show that the only continuously differentiable solutions of \( xu_x + yu_y = 0 \) on the entire plane \( \mathbb{R}^2 \) are constant functions. [Hint: Show that for any fixed \((x, y) \in \mathbb{R}^2\), the function \( g(t) = u(xt, yt) \) is constant in \( t \).]
Solution. Taking the hint, we have
\[ g'(t) = xu_x(xt, yt) + yu_y(xt, yt) = 0. \]
Hence \( g \) is constant and
\[ u(0, 0) = g(0) = g(1) = u(x, y). \]
Since this holds for any \((x, y)\), we have that \( u \) is constant and equal to \( u(0, 0) \) everywhere in \( \mathbb{R}^2 \).

5. (a) Find a solution of \( u_xu_y = 1 \) on \( \mathbb{R}^2 \) of the form \( u(x, y) = f(x) + g(y) \).

Solution. Using the form \( u(x, y) = f(x) + g(y) \) we have \( u_x(x, y) = f'(x) \) and \( u_y(x, y) = g'(y) \). Therefore
\[ f'(x)g'(y) = u_x(x, y)u_y(x, y) = 1. \]
Writing this in the form \( f'(x) = 1/g'(y) \), we see that both sides must be constant. In fact, we can simply differentiate both sides in \( x \) to find \( f''(x) = 0 \), hence \( f' \) is constant. Therefore there exists \( \rho \neq 0 \) such that
\[ f'(x) = \rho = \frac{1}{g'(y)}. \]
It follows that \( f(x) = \rho x + a \) and \( g(y) = \frac{1}{\rho} y + b \). The general solution in the form \( u(x, y) = f(x) + g(y) \) is
\[ u(x, y) = \rho x + \frac{1}{\rho} y + C, \]
where \( C = a + b \) and \( \rho \neq 0 \) are arbitrary constants. It is enough to find just one solution to get full credit.

(b) Find two different solutions of \( u_xu_y = u \) in the domain \( x \geq 0 \) and \( y \geq 0 \) that satisfy \( u(x, 0) = 0 \) and \( u(0, y) = 0 \) for all \( x \geq 0 \) and \( y \geq 0 \). [Hint: One is trivial. For the other, look for a solution in the separable form \( u(x, y) = f(x)g(y) \).]

Solution. The trivial solution is, by inspection, \( u(x, y) = 0 \). For the other, we look for a solution in the form \( u(x, y) = f(x)g(y) \). Then
\[ f'(x)g(y)f(x)g'(y) = u_x(x,y)u_y(x,y) = u(x,y) = f(x)g(y). \]
Therefore
\[ f'(x)g'(y) = 1. \]
This is the same equation as in part (a). Therefore
\[ f(x) = \rho x + a \quad \text{and} \quad g(y) = \frac{1}{\rho} y + b, \]
and so the general solution is
\[ u(x, y) = f(x)g(y) = xy + \rho bx + \frac{a}{\rho}y + ab. \]

Since \( u(x, 0) = 0 \) we must have \( \rho bx + ab = 0 \) for all \( x \). Differentiating in \( x \) yields \( \rho b = 0 \). Since \( \rho \neq 0 \) we must have \( b = 0 \). Using the other boundary condition \( u(0, y) = 0 \) yields \( \frac{a}{\rho}y = 0 \) for all \( y \). Hence \( a = 0 \) as well, and we get the solution \( u(x, y) = xy \). \( \square \)

6. (a) Write down a formula for the general solution to the nonlinear PDE \( u_t + u_x + u^2 = 0 \).

Solution. Let \( f(x) = u(x_0, 0) \) and fix \( x_0 \). As for question 3, the left hand side involves the directional derivative of \( u \) in the direction \((1, 1)\). Hence, we define, as in problem 3, \( g(t) = u(x_0 + t, t) \), and compute
\[ g'(t) = u_t(x_0 + t, t) + u_x(x_0 + t, t) = -u(x_0 + t, t)^2 = -g(t)^2. \]
Thus \( g'(t) = -g(t)^2 \). We can solve this ODE via separation of variables as follows:
\[ \frac{d}{dt} \left( \frac{1}{g(t)} \right) = -\frac{g'(t)}{g(t)^2} = 1. \]
Therefore \( \frac{1}{g(t)} = t + C \),
\[ g(t) = \frac{1}{t + C}. \]
for an arbitrary constant \( C \). Since
\[ \frac{1}{C} = g(0) = u(x_0, 0) = f(x_0), \]
we have \( C = 1/f(x_0) \). Therefore
\[ u(x_0 + t, t) = g(t) = \frac{1}{t + \frac{1}{f(x_0)}} = \frac{f(x_0)}{tf(x_0) + 1}. \] (1)
We want the solution at \((x, t)\) so set \( x_0 + t = x\) (or \( x_0 = x - t \)) to find that
\[ u(x, t) = \frac{f(x - t)}{tf(x - t) + 1}. \]
This is the general solution, for an arbitrary function \( f(x) = u(x, 0) \). \( \square \)

(b) Show that if the initial data \( f(x) = u(x, 0) \) is nonnegative and bounded \( 0 \leq f(x) \leq M \), then the solution exists for all \( t > 0 \), and \( u(x, t) \to 0 \) as \( t \to \infty \).
Solution. The denominator vanishes when \( t = -f(x - t)^{-1} \leq 0 \), so the solution \( u(x, t) \) exists for all \( t > 0 \). Furthermore, if \( f(x - t) = 0 \) then \( u(x, t) = 0 \). If \( f(x - t) > 0 \) then
\[
u(x, t) = \frac{1}{t + \frac{1}{f(x-t)}} \leq \frac{1}{t}.
\]
Therefore
\[
0 \leq u(x, t) \leq \frac{1}{t} \quad \text{for all } t > 0.
\]
It follows that \( \lim_{t \to \infty} u(x, t) = 0 \) (uniformly in \( x \) in fact).

(c) On the other hand, if the initial data \( f(x) \) is negative at some \( x \), show that the solution blows up in finite time: That is \( \lim_{t \to \tau^-} u(y, t) = -\infty \) for some \( \tau > 0 \) and \( y \in \mathbb{R} \).

Solution. Suppose that \( f(x_0) < 0 \) for some \( x_0 \). Let us write the solution in the form (1)
\[
u(x_0 + t, t) = \frac{f(x_0)}{tf(x_0) + 1} = \frac{1}{t - \tau},
\]
where \( \tau = -f(x_0)^{-1} > 0 \). The solution clearly blows up at time \( t = \tau \) and \( y = x_0 + \tau \).

(d) Find a formula for the earliest blow-up time \( \tau_* > 0 \).

Solution. By part (c), the solution blows up at \( \tau = -f(x)^{-1} \), provided \( f(x) < 0 \). Since \( f \) may be negative at many different \( x \), the earliest blow-up time is the minimum of all blow-up times, hence
\[
\tau_* = \min\{-f(x)^{-1} : x \in \mathbb{R} \text{ and } f(x) < 0\}.
\]
In other words, the larger the initial data \( f(x) \) (negatively), the sooner the solution will blow up.