MATH 5587 – HOMEWORK 5 (DUE THURSDAY OCT 13)

In any question that asks for a series representation for the solution of a PDE, you do not need to give formulas for the coefficients A_n and B_n.

1. Let $\varphi(x) = x^2$ for $0 \le x \le 1 = \ell$.
   (a) Calculate the Fourier sine series for φ.
   (b) Calculate the Fourier cosine series for φ.
2. Find the Fourier cosine series of the function $\sin(x)$ in the interval $(0, \pi)$. Use it to compute the sums
$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$
3. (a) Use the Fourier expansion to explain why the note produced by a violin string rises by one octave when the string is clamped exactly at its midpoint. [Each increase by an octave corresponds to a doubling of the frequency.]
   (b) Explain why the pitch of the note rises when the string is tightened.
4. A quantum-mechanical particle on the line with an infinite potential outside the interval $(0, \ell)$ ("particle in a box") is described by Schrödinger's equation
$$u_t = i u_{xx} \quad \text{on } (0, \ell),$$
where $i = \sqrt{-1}$, with Dirichlet conditions $u(0,t) = u(\ell, t) = 0$ at the ends. Use separation of variables to find a representation formula for $u(x,t)$ as a series.
5. Consider waves in a resistant medium, which satisfy the equation
$$\text{(W)} \quad \begin{cases} u_{tt} - c^2 u_{xx} + r u_t = 0, & 0 < x < \ell,\ t > 0 \\ u(x,0) = \varphi(x), & 0 < x < \ell \\ u_t(x,0) = \psi(x), & 0 < x < \ell \\ u(0,t) = u(\ell,t) = 0 & t > 0, \end{cases}$$
where r is a constant, $0 < r < 2\pi c/\ell$. Use separation of variables to write down the series expansion of the solution.
6. Consider the wave equation $u_{tt} = c^2 u_{xx}$ for $0 < x < \ell$ with boundary conditions $u_x(0,t) = u(\ell,t) = 0$.
   (a) Show that the eigenfunctions are $X_n(x) = \cos\left((n + \frac{1}{2})\pi x/\ell\right)$.
   (b) Write down the series expansion for a solution $u(x,t)$.
7. Consider diffusion inside an enclosed circular tube. Let its length (circumference) be 2ℓ. Let x denote the arc length parameter where $-\ell \le x \le \ell$. Then the concentration of the diffusing substance satisfies
$$\begin{cases} u_t - k u_{xx} = 0, & -\ell < x < \ell,\ t > 0 \\ u(-\ell,t) = u(\ell,t) & t > 0 \\ u_x(-\ell,t) = u_x(\ell,t) & t > 0. \end{cases}$$

These are called **periodic boundary conditions**.

(a) Show that the eigenvalues are $\lambda = (n\pi/\ell)^2$ for $n = 0, 1, 2, 3, \ldots$.

(b) Show that the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{\pi n x}{\ell}\right) + B_n \sin\left(\frac{\pi n x}{\ell}\right)\right) \exp\left(\frac{-n^2\pi^2 k t}{\ell^2}\right).$$