Math 5587 – Homework 6 (Due Thursday Oct 20)

1. For each of the following functions, state whether it is even, odd, or neither, and whether it is periodic. If periodic, what is the smallest period?

   (a) $\sin(ax)$ for $a > 0$
   (b) e^{ax} for $a > 0$
   (c) x^m for an integer m
   (d) $\tan(x^2)$
   (e) $|\sin(x/b)|$ for $b > 0$
   (f) $x\cos(ax)$ for $a > 0$

2. Let $f(x)$ be 2π periodic. Show that

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{a-\pi}^{a+\pi} f(x)\,dx$$

   for all real numbers a. That is, the integral of a 2π-periodic function is the same over any interval of length 2π. [Hint: Show that there exists an integer n such that

$$a - \pi \le 2\pi n - \pi < a + \pi.$$

   Write the integral as

$$\int_{a-\pi}^{a+\pi} f(x)\,dx = \int_{a-\pi}^{2\pi n-\pi} f(x)\,dx + \int_{2\pi n-\pi}^{a+\pi} f(x)\,dx,$$

   and use the periodicity of f to complete the proof from here.]

3. Recall a function f is Lipschitz if there exists $L > 0$ such that

$$|f(x) - f(y)| \le L|x - y| \quad \text{for all } x, y$$

   (a) Show that every Lipschitz function f is continuous. [Hint: A function is continuous if $\lim_{y\to x} f(y) = f(x)$ for all x.]
   (b) Show that if f is continuously differentiable and f' is bounded, then f is Lipschitz. [Hint: First assume $x > y$ and recall the fundamental theorem of calculus:

$$f(x) - f(y) = \int_y^x f'(s)\,ds.$$

   Take absolute values of both sides, and use the inequality $|\int_a^b g(s)\,ds| \le \int_a^b |g(s)|\,ds$, which holds for $a < b$. Then repeat a similar argument when $y > x$. Recall f' is bounded means there exists $L > 0$ such that $|f'(x)| \le L$ for all x.]

4. Consider the geometric series $\sum_{n=0}^{\infty}(-1)^n x^{2n}$.

   (a) Does it converge pointwise in the interval $-1 < x < 1$?

(b) Does it converge uniformly in the interval $-1 < x < 1$?

(c) Does it converge in the L^2 sense (i.e., in norm) in the interval $-1 < x < 1$? [Hint: You can compute the partial sums explicitly.]

5. Prove the Cauchy-Schwarz inequality

$$|(f, g)| \leq \|f\|\|g\|,$$

for any pair of functions f and g on an interval (a, b). [Hint: Consider the expression $h(t) := \|f + tg\|^2$ where $t \in \mathbb{R}$, and find the value of t that minimizes h.]

6. Prove the Cauchy-Schwarz inequality for infinite series

$$\sum_{n=1}^{\infty} a_n b_n \leq \left(\sum_{n=1}^{\infty} a_n^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}}.$$

[Hint: Write $(a, b) = \sum_{n=1}^{N} a_n b_n$ and $\|a\|^2 = (a, a)$, and use an argument similar to that for the previous question. Prove it first for finite sums, and then pass to the limit.]

7. Show that if f is a continuously differentiable 2π-periodic function satisfying

$$\int_{-\pi}^{\pi} f(x)\, dx = 0, \tag{1}$$

then we have

$$\int_{-\pi}^{\pi} f(x)^2\, dx \leq \int_{-\pi}^{\pi} f'(x)^2\, dx. \tag{2}$$

The inequality above is called a Poincaré inequality. Give an example showing that (2) may not hold when the zero mean condition (1) fails. [Hint: Recall that since f is continuously differentiable and 2π-periodic, the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \text{where } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

converges uniformly, and hence it also converges in norm. Therefore Plancheral's identity

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)^2\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

holds. Use integration by parts to find a relationship between the Fourier coefficients c_n of f, and the coefficients

$$d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx}\, dx$$

of f'. Then apply Bessel's inequality. Make sure to point out where you use the zero mean condition (1).]