Math 5587 – Homework 6 (Due Thursday Oct 20)

1. For each of the following functions, state whether it is even, odd, or neither, and whether
it is periodic. If periodic, what is the smallest period?

(a) \( \sin(ax) \) for \( a > 0 \) Periodic with period \( \frac{2\pi}{a} \)
(b) \( e^{ax} \) for \( a > 0 \) Not periodic
(c) \( x^m \) for an integer \( m \) Not periodic unless \( m = 0 \). In this case there is no smallest
period.
(d) \( \tan(x^2) \) Not periodic
(e) \( |\sin(x/b)| \) for \( b > 0 \) Periodic with period \( b\pi \).
(f) \( x \cos(ax) \) for \( a > 0 \) Not periodic

2. Let \( f(x) \) be \( 2\pi \) periodic. Show that

\[
\int_{-\pi}^{\pi} f(x) \, dx = \int_{a-\pi}^{a+\pi} f(x) \, dx
\]

for all real numbers \( a \). That is, the integral of a \( 2\pi \)-periodic function is the same over
any interval of length \( 2\pi \). [Hint: Show that there exists an integer \( n \) such that
\( a - \pi \leq 2\pi n - \pi < a + \pi \).

Write the integral as

\[
\int_{a-\pi}^{a+\pi} f(x) \, dx = \int_{a-\pi}^{2\pi n - \pi} f(x) \, dx + \int_{2\pi n - \pi}^{a+\pi} f(x) \, dx,
\]

and use the periodicity of \( f \) to complete the proof from here.]

Solution. Let \( n \) be the smallest integer such that \( 2\pi n \geq a \). Then \( 2\pi(n + 1) < a \) and we have
\( 2\pi n = 2\pi(n - 1) + 2\pi < a + 2\pi \).

Therefore

\( a - \pi \leq 2\pi n - \pi < a + \pi \),

or

\( a - \pi \leq 2\pi n - \pi < a + \pi \).

Write the integral as

\[
\int_{a-\pi}^{a+\pi} f(x) \, dx = \int_{a-\pi}^{2\pi n - \pi} f(x) \, dx + \int_{2\pi n - \pi}^{a+\pi} f(x) \, dx.
\]

In the first integral, make the substitution \( y = x + 2\pi \). Then since \( f(y) = f(x + 2\pi) = f(x) \) we have
\[
\int_{a-\pi}^{a+\pi} f(x) \, dx = \int_{a+\pi}^{2\pi n + \pi} f(y) \, dy + \int_{2\pi n - \pi}^{a+\pi} f(x) \, dx.
\]
Relabeling $y = x$ in the first integral we can combine each of the terms above to find that
\[
\int_{a-atop\pi}^{a+\pi} f(x) \, dx = \int_{2\pi a-atop\pi}^{2\pi a+\pi} f(x) \, dx.
\]
Now make the substitution $y = x - 2\pi n$. Then since $f(y) = f(x - 2\pi n) = f(x)$ we have
\[
\int_{a-atop\pi}^{a+\pi} f(x) \, dx = \int_{-\pi}^{\pi} f(y) \, dy. \tag*{\square}
\]

3. Recall a function $f$ is Lipschitz if there exists $L > 0$ such that
\[
|f(x) - f(y)| \leq L|x - y| \text{ for all } x, y.
\]
(a) Show that every Lipschitz function $f$ is continuous. [Hint: A function is continuous if $\lim_{y \to x} f(y) = f(x)$ for all $x$.]

Solution. Fix $x$. Since $\lim_{y \to x} |x - y| = 0$ and
\[
0 \leq |f(x) - f(y)| \leq L|x - y|,
\]
we can use the Squeeze Lemma to find that
\[
\lim_{y \to x} |f(x) - f(y)| = 0,
\]
which is equivalent to $\lim_{y \to x} f(y) = f(x)$. Therefore $f$ is continuous. \tag*{\square}

(b) Show that if $f$ is continuously differentiable and $f'$ is bounded, then $f$ is Lipschitz. [Hint: First assume $x > y$ and recall the fundamental theorem of calculus:
\[
f(x) - f(y) = \int_{y}^{x} f'(s) \, ds.
\]
Take absolute values of both sides, and use the inequality $|\int_{a}^{b} g(s) \, ds| \leq \int_{a}^{b} |g(s)| \, ds$, which holds for $a < b$. Then repeat a similar argument when $y > x$. Recall $f'$ is bounded means there exists $L > 0$ such that $|f'(x)| \leq L$ for all $x$.]

Solution. Since $f'$ is bounded, there exists $L > 0$ such that $|f'(s)| \leq L$ for all $s$. When $x > y$ we have
\[
|f(x) - f(y)| = \left| \int_{y}^{x} f'(s) \, ds \right| \leq \int_{y}^{x} |f'(s)| \, ds \leq L(y - x) = L|x - y|.
\]
When $y > x$ we have
\[
|f(x) - f(y)| = \left| \int_{x}^{y} f'(s) \, ds \right| \leq \int_{x}^{y} |f'(s)| \, ds \leq L|x - y|.
\tag*{\square}

4. Consider the geometric series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$. 

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(a) Does it converge pointwise in the interval $-1 < x < 1$?

**Solution.** The geometric series partial sums are

$$
\sum_{n=0}^{N} (-x^2)^n = \frac{1 - (-x^2)^{N+1}}{1 + x^2} = \frac{1}{1 + x^2} - \frac{(-1)^N x^{2N+2}}{1 + x^2}.
$$

For $-1 < x < 1$, $\lim_{N \to \infty} x^{2N+2} = 0$, therefore

$$
\lim_{N \to \infty} \sum_{n=0}^{N} (-x^2)^n = \frac{1}{1 + x^2}, \quad -1 < x < 1.
$$

Therefore the series converges pointwise to $1/(1 + x^2)$ on $-1 < x < 1$. \(\square\)

(b) Does it converge uniformly in the interval $-1 < x < 1$?

**Solution.** The series does not converge uniformly. To see this, write

$$
\left| \sum_{n=0}^{N} (-x^2)^n - \frac{1}{1 + x^2} \right| = \frac{x^{2N+2}}{1 + x^2} \geq \frac{1}{2} x^{2N+2},
$$

for $-1 < x < 1$. Set $x_N = (\frac{1}{2})^{\frac{1}{N+2}}$. Then

$$
\max_{-1 < x < 1} \left| \sum_{n=0}^{N} (-x^2)^n - \frac{1}{1 + x^2} \right| \geq \sum_{n=0}^{N} (-x_N^2)^n - \frac{1}{1 + x_N^2} \geq \frac{1}{4},
$$

which does not converge to zero as $N \to \infty$. \(\square\)

(c) Does it converge in the $L^2$ sense (i.e., in norm) in the interval $-1 < x < 1$? [Hint: You can compute the partial sums explicitly.]

**Solution.** The series does converge in $L^2$. Notice that

$$
\left| \sum_{n=0}^{N} (-x^2)^n - \frac{1}{1 + x^2} \right| = \frac{x^{2N+2}}{1 + x^2} \leq x^{2N+2}.
$$

Therefore

$$
\left\| \sum_{n=0}^{N} (-x^2)^n - \frac{1}{1 + x^2} \right\|^2 \leq \int_{-1}^{1} x^{4N+4} \, dx = \frac{2}{4N+5}.
$$

Therefore

$$
\lim_{N \to \infty} \left\| \sum_{n=0}^{N} (-x^2)^n - \frac{1}{1 + x^2} \right\| = 0. \quad \square
$$

5. Prove the Cauchy-Schwarz inequality

$$
|\langle f, g \rangle| \leq \|f\| \|g\|;
$$

for any pair of functions $f$ and $g$ on an interval $(a, b)$. [Hint: Consider the expression $h(t) := \|f + tg\|^2$ where $t \in \mathbb{R}$, and find the value of $t$ that minimizes $h$.]
Proof. We may assume that \( \|f\| < \infty \) and \( \|g\| < \infty \), otherwise the Cauchy-Schwarz inequality is trivial. Write \( h(t) = \|f + tg\|^2 \) and note that
\[
h(t) = \int_a^b (f + tg)^2 \, dx = \int_a^b f^2 + 2tfg + t^2g^2 \, dx = \int_a^b f^2 \, dx + 2t\int_a^b fg \, dx + t^2\int_a^b g^2 \, dx.
\]
Therefore
\[
h(t) = \|f\|^2 + 2t(f, g) + t^2\|g\|^2,
\]
Assume first that \( \|g\| = 0 \). Then set \( t = -\frac{1}{2}(f, g) \) for \( A > 0 \) to find that
\[
0 \leq h(t) = \|f\|^2 - A(f, g)^2.
\]
Therefore
\[
(f, g)^2 \leq \frac{1}{A}\|f\|^2.
\]
Sending \( A \to \infty \) yields \((f, g)^2 = 0 = \|f\|^2\|g\|^2\).
Assume now that \( \|g\| \neq 0 \) and compute
\[
h'(t) = 2(f, g) + 2t\|g\|^2.
\]
Then \( h'(t) = 0 \) if and only if \( t = -\frac{1}{\|g\|^2}(f, g) \). Substituting this into \( h \) yields
\[
0 \leq h(t) = \|f\|^2 - 2\frac{(f, g)^2}{\|g\|^2} + \frac{(f, g)^2}{\|g\|^2}.
\]
Rearranging yields
\[
(f, g)^2 \leq \|f\|^2\|g\|^2.
\]

6. Prove the Cauchy-Schwarz inequality for infinite series
\[
\sum_{n=1}^{\infty} a_n b_n \leq \left( \sum_{n=1}^{\infty} a_n^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} b_n^2 \right)^{1/2}.
\]

[Hint: Write \((a, b) = \sum_{n=1}^{N} a_n b_n \) and \( \|a\|^2 = (a, a) \), and use an argument similar to that for the previous question. Prove it first for finite sums, and then pass to the limit.]

Proof. The proof is similar to question 5, so we will just sketch it here. We may assume that \( \sum a_n^2 < \infty \) and \( \sum b_n^2 < \infty \), otherwise the result is trivial.
We first claim that \( \sum_{n=1}^{\infty} a_n b_n \) converges. To see this we use the Cauchy inequality
\[
|a_n b_n| \leq \frac{1}{2}(a_n^2 + b_n^2).
\]
To derive the inequality above, write \((|a_n| - |b_n|)^2 \geq 0\), expand and rearrange. By comparison, \( \sum a_n b_n \) converges absolutely, and hence is convergent.
Fix \( N \) and use an argument similar to problem 7 to find that
\[
\sum_{n=1}^{N} a_n b_n = (a, b) \leq \|a\| \|b\| = \left( \sum_{n=1}^{N} a_n^2 \right)^{1/2} \left( \sum_{n=1}^{N} b_n^2 \right)^{1/2}.
\]
Sending \( N \to \infty \) completes the proof. \( \square \)
7. Show that if \( f \) is a continuously differentiable \( 2\pi \)-periodic function satisfying

\[
\int_{-\pi}^{\pi} f(x) \, dx = 0,
\]  

then we have

\[
\int_{-\pi}^{\pi} f(x)^2 \, dx \leq \int_{-\pi}^{\pi} f'(x)^2 \, dx.
\]  

The inequality above is called a Poincaré inequality. Give an example showing that (2) may not hold when the zero mean condition (1) fails. [Hint: Recall that since \( f \) is continuously differentiable and \( 2\pi \)-periodic, the Fourier series

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \text{where } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx
\]

converges uniformly, and hence it also converges in norm. Therefore Plancheral’s identity

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2
\]

holds. Use integration by parts to find a relationship between the Fourier coefficients \( c_n \) of \( f \), and the coefficients

\[
d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x)e^{-inx} \, dx
\]

of \( f' \). Then apply Bessel’s inequality. Make sure to point out where you use the zero mean condition (1).

**Solution.** Since \( f \) is continuously differentiable and \( 2\pi \)-periodic, the Fourier series

\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \text{where } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx
\]

converges uniformly, and hence it also converges in norm. Therefore Plancheral’s identity

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2
\]

holds. Using integration by parts we have

\[
d_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x)e^{-inx} \, dx
\]

\[
= \frac{1}{2\pi} f(x)e^{-inx}\bigg|_{-\pi}^{\pi} + \frac{in}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} \, dx
\]

\[
= \frac{-i}{\pi} f(\pi) \sin(n\pi) + inc_n = inc_n,
\]

for \( n \neq 0 \). Therefore \(|d_n| = n|c_n| \) for \( n \neq 0 \). For \( n = 0 \) we have

\[
d_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) \, dx = \frac{1}{2\pi} (f(\pi) - f(-\pi)) = 0,
\]
and

\[ c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = 0 \]

due to the zero mean condition (1). Therefore

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = |c_{n=0}|^2 = \sum_{n=-\infty}^{\infty} \frac{|d_n|^2}{n^2} \leq \sum_{n=-\infty}^{\infty} |d_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x)^2 \, dx,
\]

where the last line follows from Bessel’s inequality.

The Poincaré inequality (2) is not true when the zero mean condition (1) fails to hold. For instance, if \( f(x) = 1 \) for all \( x \), then

\[
\int_{-\pi}^{\pi} f(x)^2 = 2\pi > 0 = \int_{-\pi}^{\pi} f'(x)^2 \, dx.
\]

\( \Box \)