Math 5587 – Homework 7 Solutions

1. Consider the full Fourier series on the interval $-\ell \leq x \leq \ell$ for a function $f(x)$

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right) + B_n \sin\left(\frac{n\pi x}{\ell}\right),$$

where

$$A_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) \, dx \quad \text{and} \quad B_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) \, dx.$$

(a) Assume that $f$ is an even function. Show that

$$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right),$$

where

$$A_n = \frac{2}{\ell} \int_{0}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) \, dx.$$

Solution. If $f$ is even, then $f(x) \sin\left(\frac{n\pi x}{\ell}\right)$ is an odd function, and so $B_n = 0$. Likewise, $f(x) \cos\left(\frac{n\pi x}{\ell}\right)$ is an even function and so

$$A_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) \, dx = \frac{2}{\ell} \int_{0}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) \, dx. \quad \square$$

(b) Assume that $f$ is an odd function. Show that

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{\ell}\right),$$

where

$$B_n = \frac{2}{\ell} \int_{0}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) \, dx.$$

Solution. If $f$ is odd, then $f(x) \cos\left(\frac{n\pi x}{\ell}\right)$ is an odd function, and so $A_n = 0$. Likewise, $f(x) \sin\left(\frac{n\pi x}{\ell}\right)$ is an even function and so

$$B_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) \, dx = \frac{2}{\ell} \int_{0}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) \, dx. \quad \square$$
2. Let \( f \) be a \( 2\pi \)-periodic \( k \)-times continuously differentiable function and consider the complex version of the Fourier series
\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx.
\]
Recall that for complex-valued functions \( f \) and \( g \) on the interval \([-\pi, \pi]\) we define
\[
(f, g) = \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx \quad \text{and} \quad \|f\| = (f, f)^{\frac{1}{2}}.
\]
(a) Let \( g \) be any \( 2\pi \)-periodic complex-valued differentiable function. Use integration by parts to show that
\[
(f, g') = -(f', g).
\]
[Hint: Differentiation and complex conjugation commute, so \( \frac{d}{dx}(\overline{g(x)}) = \overline{g'(x)} \).]

Solution. Integrating by parts we have
\[
(f, g') = \int_{-\pi}^{\pi} f(x) \frac{d}{dx} \overline{g(x)} \, dx
\]
\[
= \int_{-\pi}^{\pi} f(x) \frac{d}{dx} g(x) \, dx
\]
\[
= \left[ f(x) \overline{g(x)} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x) \overline{g(x)} \, dx
\]
\[
= -(f', g).
\]
Note the boundary term \( f(x) \overline{g(x)} \bigg|_{-\pi}^{\pi} \) vanished because \( f \) and \( g \) are \( 2\pi \)-periodic.

(b) Use part (a) to show that
\[
c_n = \frac{1}{2\pi (in)^k} \left( f^{(k)}, e^{inx} \right),
\]
where \( f^{(k)} \) denotes the \( k \)th derivative of \( f \). [Hint: Start by using part (a) to show that
\[
c_n = \frac{1}{2\pi in} \left( f', e^{inx} \right).
\]
Then repeat the argument \( k \) times (or use induction).]

Proof. We will use induction. For the base case, use \( g(x) = \frac{1}{in} e^{inx} \) in part (a). Then \( g'(x) = e^{inx} \) and so
\[
c_n = \frac{1}{2\pi} (f, g') = -\frac{1}{2\pi} (f', \frac{1}{in} e^{inx}) = \frac{1}{2\pi in} (f', e^{inx}).
\]
Notice we used the identity \( (f, ag) = \overline{a}(f, g) \) for any complex number \( a \). This establishes the base case.
Now suppose that
\[ c_n = \frac{1}{2\pi (in)^{k-1}} \left( f^{(k-1)}, e^{inx} \right) \]
for \( k \geq 2 \). Using the same argument as in the base case we have
\[ c_n = \frac{1}{2\pi (in)^{k-1}} \left( f^{(k-1)}, e^{inx} \right) = \frac{1}{2\pi (in)^{k}} \left( f^{(k)}, e^{inx} \right). \]
The proof is completed by mathematical induction. \( \square \)

(c) Use part (b) and the Cauchy-Schwarz inequality (HW 6) to show that for any \( k \geq 1 \)
\[ |c_n| \leq \frac{L_k}{n^k} \]
where \( L_k = \frac{1}{\sqrt{2\pi}} \| f^{(k)} \|. \)
That is, if \( f \) is \( k \)-times continuously differentiable then the Fourier coefficients \( c_n \) of \( f \) decay to zero at least as fast as \( 1/n^k \).

**Solution.** Recall that \( \|e^{inx}\|^2 = 2\pi \). By Cauchy-Schwarz we have
\[ |c_n| = \frac{1}{2\pi n^k} |(f^{(k)}, e^{inx})| \leq \frac{1}{2\pi n^k} \|f^{(k)}\| \|e^{inx}\| = \frac{1}{\sqrt{2\pi n^k}} \|f^{(k)}\|. \] \( \square \)

3. Let \( X(x) \) be a solution of the eigenvalue problem
\[ X'' + \lambda X(x) = 0 \quad \text{for} \quad a \leq x \leq b, \]
where \( \lambda \) is a real number.

(a) Assume that
\[ X X' \bigg|_a^b = X(b)X'(b) - X(a)X'(a) \leq 0. \] \( (1) \)
Show that if \( X \) is not the zero function, then \( \lambda \geq 0 \). Hence \( (1) \) guarantees there are no negative eigenvalues. [Hint: Show that \( \lambda \|X\|^2 = \lambda (X, X) = -(X'', X) \). Then integrate by parts and use \( (1) \) to show that \( \lambda \|X\|^2 \geq 0 \).]

**Proof.** As in the hint, we write
\[ \lambda \|X\|^2 = (\lambda X, X) = -(X'', X) = - \int_a^b X''(x)X(x) \, dx. \]
Integrating by parts we have
\[ \lambda \|X\|^2 = -X'X \bigg|_a^b + \int_a^b X'(x)^2 \, dx. \]
By \( (1) \), \( -X'X \bigg|_a^b \geq 0 \). Therefore
\[ \lambda \|X\|^2 \geq \int_a^b X'(x)^2 \, dx \geq 0, \]
where the last inequality is true because we are integrating \( X'(x)^2 \), which is positive. So, provided \( \|X\| \neq 0 \) (i.e., \( X \) is not the zero function), then \( \lambda \geq 0 \). \( \square \)
(b) Show that (1) holds when $X$ satisfies

i. Homogeneous Dirichlet boundary conditions $X(a) = X(b) = 0$,

ii. Homogeneous Neumann boundary conditions $X'(a) = X'(b) = 0$,

iii. Mixed boundary conditions $X'(a) = X(b) = 0$ or $X(a) = X'(b) = 0$,

iv. Periodic boundary conditions $X(a) = X(b)$ and $X'(a) = X'(b)$.

**Solution.** These are all direct calculations.

4. Let $X_1(x)$ and $X_2(x)$ be solutions of the eigenvalue problems

$$X_1'' + \lambda_1 X_1(x) = 0 \quad \text{for} \quad a \leq x \leq b,$$

and

$$X_2'' + \lambda_2 X_2(x) = 0 \quad \text{for} \quad a \leq x \leq b,$$

for different eigenvalues $\lambda_1 \neq \lambda_2$.

(a) Show that

$$-X_1'' X_2 + X_1 X_2'' = \frac{d}{dx}(-X_1' X_2 + X_1' X_2').$$

[Hint: Use product rule to expand the right hand side.]

**Solution.** We have

$$\frac{d}{dx}(-X_1' X_2 + X_1' X_2') = -X_1'' X_2 - X_1' X_2' + X_1' X_2' + X_1 X_2'' = -X_1'' X_2 + X_1 X_2''.$$

(b) Use part (a) and the fundamental theorem of calculus to show that

$$(\lambda_1 - \lambda_2)(X_1, X_2) = (-X_1' X_2 + X_1' X_2') \bigg|_a^b.$$

**Solution.** By the fact that $X_1$ and $X_2$ solve the eigenvalue problems above

$$-X_1'' X_2 + X_1 X_2'' = \lambda_1 X_1 X_2 - \lambda_2 X_1 X_2 = (\lambda_1 - \lambda_2)X_1 X_2.$$

Combining this with part (b) and the fundamental theorem of calculus we have

$$(\lambda_1 - \lambda_2)(X_1, X_2) = \int_a^b (\lambda_1 - \lambda_2)X_1 X_2 \, dx = \int_a^b \frac{d}{dx}(-X_1' X_2 + X_1' X_2') \, dx = (-X_1' X_2 + X_1' X_2') \bigg|_a^b.$$

(c) Conclude from part (b) that $X_1$ and $X_2$ are orthogonal (i.e., $(X_1, X_2) = 0$) if and only if

$$(-X_1' X_2 + X_1' X_2') \bigg|_a^b = 0.$$  \hfill (2)
Solution. Since $\lambda_1 \neq \lambda_2$, $\lambda_1 - \lambda_2 \neq 0$. It therefore follows from part (b) that $(X_1, X_2) = 0$ if and only if (2) holds.

(d) Boundary conditions are called symmetric whenever (2) holds. By part (b), whenever we have symmetric boundary conditions, eigenfunctions corresponding to different eigenvalues are always orthogonal. Show that each of the boundary conditions listed below are symmetric.

i. Homogeneous Dirichlet boundary conditions $X_i(a) = X_i(b) = 0$.
ii. Homogeneous Neumann boundary conditions $X_i'(a) = X_i'(b) = 0$.
iii. Periodic boundary conditions $X_i(a) = X_i(b)$ and $X_i'(a) = X_i'(b)$.
iv. Robin boundary conditions $X_i(a) - AX_i'(a) = X_i(b) + AX_i'(b) = 0$, where $A$ is a real number.

In each case, the boundary conditions are assumed to hold for both $i = 1$ and $i = 2$.

Solution. This is a straightforward calculation.

5. Solve the heat equation

\[
\begin{align*}
&\begin{cases}
  u_t - u_{xx} = 0, & \text{for } 0 < x < \pi \text{ and } t > 0 \\
  u(0, t) = 0, & \text{for } t > 0 \\
  u(\pi, t) = \pi, & \text{for } t > 0 \\
  u(x, 0) = \cos(x), & \text{for } 0 \leq x \leq \pi.
\end{cases}
\end{align*}
\]

Give a complete answer, including the values of the coefficients $B_n$ of the series. After solving for $u$, compute

\[
\lim_{t \to \infty} u(x, t).
\]

[Hint: First define $v(x, t) = u(x, t) - x$ to convert the problem into one with homogenous Dirichlet conditions $v(0, t) = v(\pi, t) = 0$. Then solve for $v$ instead.]

Solution. Let $v(x, t) = u(x, t) - x$. Then we compute that $v$ solves the heat equation

\[
\begin{align*}
&\begin{cases}
  v_t - v_{xx} = 0, & \text{for } 0 < x < \pi \text{ and } t > 0 \\
  v(0, t) = 0, & \text{for } t > 0 \\
  v(\pi, t) = 0, & \text{for } t > 0 \\
  v(x, 0) = \cos(x) - x, & \text{for } 0 \leq x \leq \pi.
\end{cases}
\end{align*}
\]

This is the Dirichlet problem for the heat equation, with general solution (obtained by separation of variables)

\[
v(x, t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin(nx).
\]

Therefore

\[
u(x, t) = x + v(x, t) = x + \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin(nx).
\]
Setting $t = 0$ we have

$$\cos(x) = u(x, 0) = x + \sum_{n=1}^{\infty} B_n \sin(nx),$$

and so

$$\cos(x) - x = \sum_{n=1}^{\infty} B_n \sin(nx),$$

This is a Fourier Sine series, and we have

$$B_n = \frac{2}{\pi} \int_{0}^{\pi} (\cos(x) - x) \sin(nx) \, dx.$$

If we split this up into two pieces we have

$$B_n = \frac{2}{\pi} \int_{0}^{\pi} \cos(x) \sin(nx) \, dx - \frac{2}{\pi} \int_{0}^{\pi} x \sin(nx) \, dx.$$

We’ve done very similar integrals in class and on previous homework. The answer is

$$B_n = \begin{cases} \frac{4n}{\pi(n^2-1)} + \frac{2}{n}, & \text{if } n \text{ even} \\ -\frac{2}{n}, & \text{if } n \text{ odd} \end{cases}$$

These coefficients along with the formula

$$u(x, t) = x + \sum_{n=1}^{\infty} B_n e^{-n^2t} \sin(nx)$$

give the complete solution of the PDE. We see now that

$$\lim_{t \to \infty} u(x, t) = x.$$