Math 5587 – Homework 8 (Due Thursday Nov 10)

In each problem, $D \subset \mathbb{R}^n$ is open and bounded, and $n = 3$ unless otherwise specified.

1. Let $u \in C^2(\overline{D})$ be a solution of

$$\left.\begin{aligned} u - \Delta u &= f \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D. \end{aligned}\right\} \tag{1}$$

   Use energy methods to show that

$$\iiint_D u^2 + \|\nabla u\|^2 \, d\mathbf{x} \leq \iiint_D f^2 \, d\mathbf{x}. \tag{2}$$

   Indicate how this is a stability estimate for (1). [Hint: Multiple the PDE by u, integrate both sides over D, and then use Green's first identity. Use Cauchy's inequality $2ab \leq a^2 + b^2$ on the right hand side.]

2. Use energy methods to prove uniqueness of solutions $u \in C^2(\overline{D})$ of the Robin problem

$$\left.\begin{aligned} -\Delta u(\mathbf{x}) &= f(\mathbf{x}) \quad \text{if } \mathbf{x} \in D \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) + a(\mathbf{x})u(\mathbf{x}) &= h(\mathbf{x}) \quad \text{if } \mathbf{x} \in \partial D, \end{aligned}\right\}$$

   provided $a > 0$.

3. Consider the heat equation

$$\begin{cases} u_t - \Delta u = f & \text{in } D \times (0, T] \\ u = g & \text{on } D \times \{t = 0\} \\ u = h & \text{on } \partial D \times (0, T]. \end{cases}$$

   Prove uniqueness using energy methods. [Hint: Let u and v be two solutions and set $w = u - v$. Define the energy

$$e(t) = \iiint_D w(\mathbf{x}, t)^2 \, d\mathbf{x}.$$

   Show that $e'(t) \leq 0$ and $e(0) = 0$. Conclude that $e(t) = 0$ for all t. You will need to use Green's identities.]

4. Let $u, v \in C^2(\overline{D})$ satisfy

$$-\Delta u + F(u) \leq -\Delta v + F(v) \quad \text{in } D,$$

   where $F : \mathbb{R} \to \mathbb{R}$ is increasing, that is, $F(s) \leq F(t)$ whenever $s \leq t$. Prove the comparison principle:

   If $u \leq v$ on ∂D then $u \leq v$ everywhere in D.

[Hint: Assume to the contrary that $u(\mathbf{x}) > v(\mathbf{x})$ for some $\mathbf{x} \in D$, and define the **open** set $D' = \{\mathbf{x} \in D \mid u(\mathbf{x}) > v(\mathbf{x})\}$. Set $w := u - v$ and show that $\Delta w \geq 0$ on D' and $w \leq 0$ on $\partial D'$. Use the weak maximum principle on D' to get a contradiction.]

5. Give an example in $n = 1$ dimension to show that the comparison principle from Problem 4 may not hold if F is not increasing. [Hint: Consider the eigenvalue problem $u''(x) + u(x) = 0$ with $u(0) = u(\pi) = 0$ that we encountered in separation of variables.]

6. Use the comparison principle from Problem 4 to prove uniqueness of solutions $u \in C^2(\overline{D})$ of

$$\left.\begin{aligned} -\Delta u + F(u) &= f \quad \text{in } D \\ u &= g \quad \text{on } \partial D, \end{aligned}\right\} \tag{3}$$

   where $F : \mathbb{R} \to \mathbb{R}$ is increasing.

7. A function $u \in C^2(\overline{D})$ is **subharmonic** in D if $-\Delta u \leq 0$ in D. Let $u(x, y)$ be subharmonic in an open set $D \subset \mathbb{R}^2$.

   (a) Show that

$$u(x, y) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x + a\cos\theta, y + a\sin\theta)\, d\theta,$$

   for all $a > 0$ such that the ball of radius a centered at (x, y) belongs to D. [Hint: It is enough to prove the result for $x = y = 0$, by translation invariance. Use Poisson's integral formula to construct a harmonic function v on the disk $x^2 + y^2 \leq a^2$ with boundary values $v = u$ for $x^2 + y^2 = a^2$. Then use the comparison principle from Problem 4 to show that $u \leq v$ in the disk.]

   (b) Integrate the expression above in polar coordinates to show that

$$u(x, y) \leq \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a u(x + r\cos\theta, y + r\sin\theta)\, r dr d\theta,$$

   whenever the ball $B(\mathbf{x}, a)$ is contained in D.

8. Let $\{\alpha_n\}_{n=1}^{\infty}$ denote the positive solutions of the equation $\alpha \tan(\alpha) = 1$. Find the solution $u(x, y)$ of the boundary-value problem

$$\begin{cases} \Delta u = 0, & 0 < x < 1, \quad y > 0 \\ u_x(0, y) = 0, & y > 0 \\ u(1, y) + u_x(1, y) = 0, & y > 0 \\ u(x, 0) = 1, & 0 < x < 1 \end{cases}$$

   that is bounded as $y \to \infty$. [Hint: Use separation of variables and look for a series solution. Use the problems from HW7 to show that the eigenvalues are nonnegative and the eigenfunctions are orthogonal. You do not need to prove that the eigenfunctions are a complete orthonormal system, but do give formulas for the coefficients of the series.]