Math 5587 – Lecture 1

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1 Where do PDE come from?

As motivation for the course, we give here some examples of PDE arising in physics and other applications.

Example 1.1 (Traffic Flow)**.** Let $u(x,t)$ be the density of cars in units of cars per mile on a highway at position x and time t. Given we know the density of traffic at time $t = 0$, i.e., we know $u(x,0)$, can we predict how traffic will flow at future times? In other words, can we deduce $u(x,t)$ for future times $t > 0$ from the initial condition $u(x,0)$? We show below that this problem reduces to solving a PDE.

Let $V(x,t)$ denote the (average) velocity in miles per hour of traffic at position x on the highway at time t. Let $F(x,t)$ denote the traffic flow at position x and time t in cars per hour. Both F and V can be measured empirically. If you stand at position x with a stopwatch and count how many cars pass by you in 1 min and multiply by 60, you get an approximation of $F(x,t)$. The fundamental equation of traffic flow is

$$\underbrace{F(x,t)}_{\text{Flow (cars/hr)}} = \underbrace{u(x,t)}_{\text{Density (cars/mile)}} \times \underbrace{V(x,t).}_{\text{Velocity (miles/hr)}} \quad . \tag{1}$$

Hence, if we know any two of the above quantities, we can compute the remaining one.

Since u is the density of traffic, we have

$$\#\text{ Cars between } x = a \text{ and } x = b \text{ is } \int_a^b u(x,t)\,dx.$$

Let us assume the highway has no on- or off-ramps. Then traffic is a conserved quantity, meaning that cars are neither created nor destroyed–they just move along the road in one direction. Therefore

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \underbrace{F(a,t) - F(b,t)}_{\text{Flow in } - \text{ Flow out}}.$$

Let's set $a = x$ and $b = x + h$ for a small positive number $h > 0$. Dividing by $h > 0$ and exchanging the derivative and integral we have

$$u_t(x,t) \approx \frac{1}{h}\int_x^{x+h} u_t(s,t)\,ds = -\frac{F(x+h,t) - F(x,t)}{h}.$$

Sending $h \to 0$ we find that

$$u_t(x,t) + F_x(x,t) = 0. \tag{2}$$

So far, we have used nothing specific to traffic flow. As such, Equation (2) holds for any conserved quantity, and is called a *mass continuity equation*.

Recalling the flow equation (1) we can write (2) as

$$u_t + (uV)_x = 0, \tag{3}$$

where we are suppressing the arguments (x,t) for convenience. If we were to expand the x partial derivative above, we would have $u_t + u_x V + uV_x = 0$, but it is more common to leave the equation in the form above.

Now, (3) has two unknowns, the traffic density $u(x,t)$ and the velocity $V(x,t)$. We evidently need another equation if we hope to solve this PDE. In general, the velocity $V(x,t)$ depends on the speed limit, driving conditions, and traffic considerations. A very simply model that works reasonably well in practice is to assume that the velocity V depends only on the density u, so

$$V(x,t) = v(u(x,t)),$$

for some function v. When u is small, the velocity $V = v(u)$ should be roughly equal to the speed limit. However, as u increases, so there is more traffic on the road, we eventually reach traffic jam conditions and the velocity of traffic decreases. So v should be a decreasing function of u. Suppose we pick such a function, say $v(u) = e^{-u}$. Then we can rewrite (3) as

$$u_t + (uv(u))_x = 0. \tag{4}$$

Again, we could expand the PDE and write it as

$$u_t + u_x v(u) + uv'(u)u_x = 0,$$

but it is more common to leave it in the form in (4). This PDE now has only a single unknown function $u(x,t)$, and so with appropriate initial and boundary conditions, we can in theory solve for the traffic density $u(x,t)$ for all future times $t > 0$. In practice, the function $v(u)$ can be determined experimentally by measuring flow and density of cars on a highway.

Equation (4) is an example of a *scalar conservation law*. If the cars are moving at a constant speed independent of u (which, as we discussed above is unrealistic), then $v(u) = c$ is constant and we obtain

$$u_t + cu_x = 0.$$

This is called a *transport* equation, since it transports the quantity u to the right with speed c.

It is important to note that although the model $V = v(u)$ is simple and useful, it has its limitations. You should think for a moment about what it means to say that the velocity $V(x,t)$ of traffic depends only on the density u at the specific point (x,t). Roughly speaking it means that drivers are not looking ahead, and will drive at the same speed until smashing into a traffic jam (a region where u is large and V is small). Perhaps this is a realistic model for some drivers, but one would hope that the vast majority are more careful. A more realistic model would be that the velocity $V(x,t)$ depends on all the values of $u(s,t)$ for $x \le s \le x+d$, where $d > 0$. That is, drivers are looking ahead, and their velocity depends on the traffic

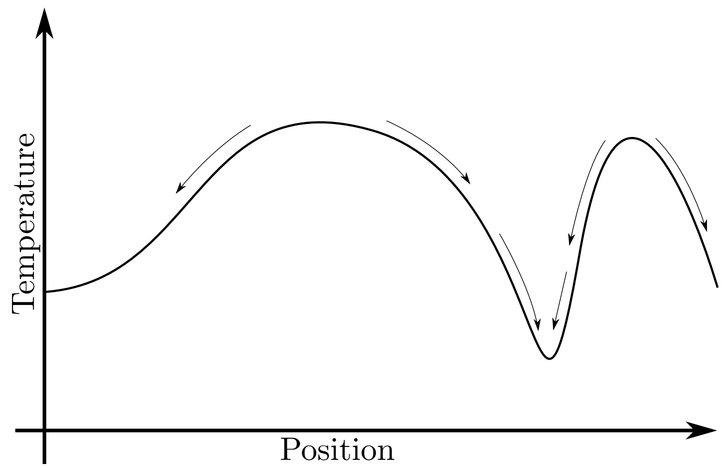

Figure 1: A depiction of heat flowing from hot to cold.

conditions in front of them up to some distance d. Such a model results in a *nonlocal* equation that is more difficult to study. Such models are actually the focus of some current research on traffic modeling.

Example 1.2 (Heat diffusion)**.** Consider the diffusion of heat along a thin insulated rod. Let $u(x,t)$ be the heat density at position x along the rod at time t. This means that $u(x,t)dx$ represents the amount of heat (usually measured in joules) contained in the rod between x and $x+dx$, and $\int_a^b u(x,t)\,dx$ represents the amount of heat between $x=a$ and $x=b$. It is generally OK to think of $u(x,t)$ as the temperature of the rod, even though this is not exactly physically accurate[1].

The question here is the following: Supposing we know the heat distribution at time $t=0$, i.e., we know $u(x,0)$, can we predict how the heat will diffuse along the rod at future times?

As in the case of traffic flow, heat is a conserved quantity–it is neither created nor destroyed. It just moves around, and we must deduce the equations governing its diffusion. Since the rod is insulated, heat can only flow to the left or right along the rod, and cannot escape in the transverse direction. Hence we are in the same situation as with traffic flow, and we have

$$u_t(x,t) + F_x(x,t) = 0, \tag{5}$$

where $F(x,t)$ denotes the heat flux at position x and time t. That is, $F(x,t)$ is the rate at which heat is flowing to the right[2]. If heat is measured in joules, then the flux is measured in joules/second, which is a watt. As in the case of traffic flow, the physical modelling requires deducing how the flux F depends on u.

As depicted in Figure 1, heat flows from hot regions into cold regions. When the slope of the graph of u is positive (i.e., $u_x > 0$), heat flows to the left, so F should be negative. When the slope u_x is negative, heat flows to the right, so F is positive. Many functions satisfy these generic properties, i.e., we could have $F = -u_x$ or $F = -u_x|u_x|$. Fortunately, physics saves the day: Fourier's law of heat conduction states that

$$F(x,t) = -k(x)u_x(x,t), \tag{6}$$

[1] The relationship between heat and temperature is material dependent.
[2] A negative rate means heat is flowing to the left

where $k(x) > 0$ is called the *thermal conductivity*. We will take this formula as a god-given identity, though it is possible to derive it from first principles, as one would do in a physics class. Substituting Fourier's law into (5) we deduce the heat (or diffusion) equation

$$u_t - (ku_x)_x = 0. \tag{7}$$

We can expand the equation and write it as

$$u_t - k_x u_x - ku_{xx} = 0,$$

although, as before, the form in (7) is more commonly used.

The thermal conductivity $k(x)$ depends on the properties of the material at location x. Small values of k indicate materials that do not conduct heat well (such as wood or cork), while larger values correspond to materials that are good conductors of heat (like metal, glass or ceramic). If the rod is composed of different materials (say half wood and half glass), then it is possible that $k = k(x)$ varies spatially along the rod. If the rod is homogeneous (composed of one material), then k is constant along the rod and the heat equation (7) reduces to

$$u_t - ku_{xx} = 0.$$

The heat equation applies in other settings as well. Consider a motionless liquid in a thin tube containing a substance (e.g., a dye). The dye moves by diffusing from high concentrations to low concentrations. Let $u(x, t)$ be the concentration of the dye at position x of the tube at time t. Fick's law of diffusion states that the rate of motion (or flux) of the dye is proportional to the negative of the concentration gradient, i.e., $F = -ku_x$ for some $k > 0$. Proceeding as above yields the same diffusion equation (7).

Example 1.3 (Vibrating String)**.** Consider a flexible elastic string of length l and let $u(x, t)$ be the displacement of the string from equilibrium at (x, t). Let $\kappa(x, t)$ denote the magnitude of the tension vector of the string at (x, t). Assuming the string is perfectly flexible, the tension vector points tangentially along the string, and has no transverse component. Let ρ be the (constant) density of the string.

Consider a small segment of the string, from x to $x + h$. See Figure 2 for reference. Since the string does not move in the horizontal (or x) direction, the horizontal components of tension acting on the segment $(x, x + h)$ must perfectly cancel out. Therefore we must have

$$\frac{\kappa(x+h,t)}{\sqrt{1+u_x(x+h,t)^2}} = \frac{\kappa(x,t)}{\sqrt{1+u_x(x,t)^2}}. \tag{8}$$

Since the string is moving in the vertical direction, the component of net force in the vertical direction must balance with mass times acceleration, according to Newton's Law $F = ma$. The acceleration is approximately $u_{tt}(x, t)$, when h is small, while the mass is ρh. Therefore we have

$$\frac{\kappa(x+h,t)u_x(x+h,t)}{\sqrt{1+u_x(x+h,t)^2}} - \frac{\kappa(x,t)u_x(x,t)}{\sqrt{1+u_x(x,t)^2}} = \rho h u_{tt}(x,t). \tag{9}$$

Now we make a key assumption. Suppose the string undergoes relatively small transverse vibrations. For instance, it could be a violin, piano, or guitar string. Therefore u_x is small, and u_x^2 is even smaller. In this case, we are justified in making the approximation

$$\sqrt{1+u_x^2} \approx 1.$$

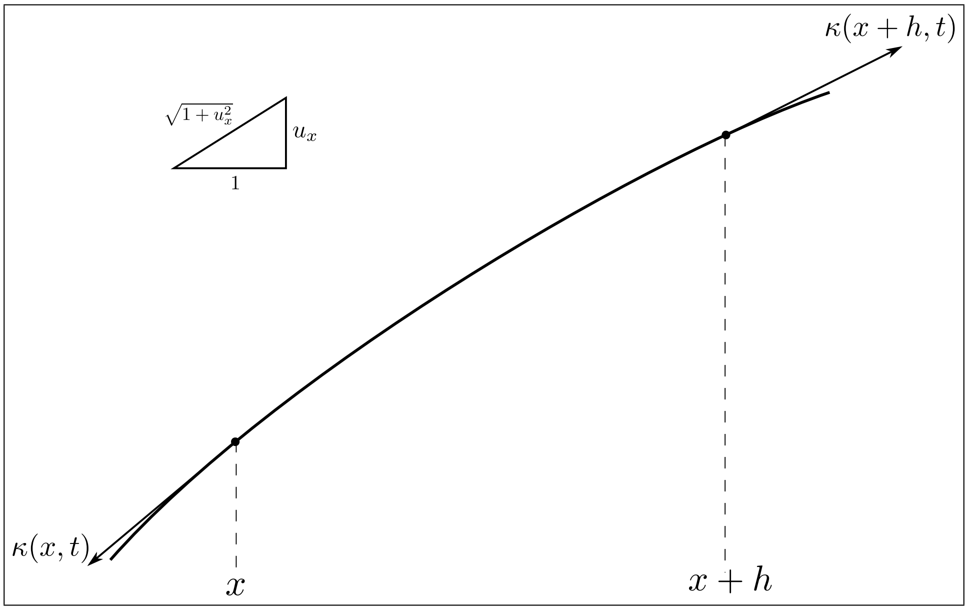

Figure 2: Balancing forces to derive the wave equation

Substituting this assumption into (8) we get that $\kappa(x+h,t) = \kappa(x,t)$, and so the magnitude of the tension vector must be constant along the string. Let κ denote the constant value of tension. Inserting this into (9) we have

$$\frac{\kappa u_x(x+h,t) - \kappa u_x(x,t)}{h} = \rho u_{tt}(x,t).$$

Sending h to zero we find that $\kappa u_{xx} = \rho u_{tt}$. This is the one-dimensional wave equation. It is common to write the wave equation in the form

$$u_{tt} - c^2 u_{xx} = 0, \tag{10}$$

where $c = \sqrt{\kappa/\rho}$. The choice of c may seem odd at first, but as it turns out c has the physical interpretation of *wave speed*.

1.1 Higher dimensional examples

Many of these examples can be extended to higher dimensions. We first review some multivariable calculus.

- Divergence: $\operatorname{div} \mathbf{v} = v^1_x + v^2_y + v^3_z$, where $\mathbf{v}(x,y,z) = (v^1(x,y,z), v^2(x,y,z), v^3(x,y,z))$.
- Divergence of gradient is the Laplace operator:

$$\operatorname{div} \nabla u = u_{xx} + u_{yy} + u_{zz} = \Delta u.$$

- Divergence theorem in 2D:

$$\iint_D \operatorname{div} \mathbf{v}\, dx\, dy = \int_{\partial D} \mathbf{v} \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the unit outward normal vector field to ∂D. Here, ∂D denotes the boundary of D.

- Divergence theorem in 3D:

$$\iiint_D \operatorname{div} \mathbf{v} \, dx \, dy \, dz = \iint_{\partial D} \mathbf{v} \cdot \mathbf{n} \, dS,$$

where $\mathbf{n}$ is the unit outward normal vector field to ∂D.

- Normal derivative:

$$\frac{\partial u}{\partial \mathbf{n}} := \nabla u \cdot \mathbf{n}.$$

The heat, or diffusion equation in higher dimensions is

$$u_t - \operatorname{div}(k \nabla u) = 0,$$

where $u = u(x, y, z, t)$, and $k(x, y, z)$ is again the thermal conductivity. When k is constant we get

$$u_t - k\Delta u = 0.$$

The wave equation in higher dimensions models a vibrating drumhead (in two dimensions) and the propagation of sound waves (in three dimensions), among many other phenomena. The wave equation in higher dimensions is

$$u_{tt} - c^2 \Delta u = 0.$$

We give brief derivations of these equations below, after a few more important examples.

Example 1.4 (Laplace and Poisson equations)**.** Consider the wave equation, or heat equation with constant thermal conductivity k. When the system is at rest (i.e., the heat has finished diffusing, or the vibrating drum has stopped moving), we have $u_t = u_{tt} = 0$, and hence

$$\Delta u = 0. \tag{11}$$

The PDE (11) is called *Laplace's equation.* We can add forcing terms to the heat and wave equations, yielding

$$u_t - \Delta u = f,$$

and

$$u_{tt} - \Delta u = f,$$

where $f = f(x, y, z)$ is a given function. Here, we have taken $c = k = 1$ in the heat and wave equations for simplicity. For the heat equation, the forcing term corresponds to injecting or removing heat at (x, y, z) at a rate of $f(x, y, z)$, while for the wave equation, f corresponds to an external force applied to the string or drumhead. If the forced wave or heat equations are at steady state then we must have

$$-\Delta u = f. \tag{12}$$

This is known as *Poisson's equation.*

Another important example is Schrödinger's equation.

Example 1.5 (Schrödinger's equation)**.** Consider a Hydrogen atom, which is a single electron orbiting a proton. Let m be the mass of the electron, e its charge, and h Planck's constant divided by 2π. Let $r = \sqrt{x^2+y^2+z^2}$ and suppose the proton is at the origin. The motion of the electron is governed by a *wave function* $u(x, y, z, t)$ which satisfies Schrödinger's equation

$$-ihu_t = \frac{h^2}{2m}\Delta u + \frac{e^2}{r}u.$$

This is usually taken as an axiom and is generally not derived from any other physical principles. Note u is complex valued. The probability of finding the electron in a region D at time t is

$$\iiint_D |u|^2 \, dx\, dy\, dz.$$

We now give short derivations of the heat and wave equations in higher dimensions.

Example 1.6 (Diffusion in higher dimensions)**.** We consider heat flow in higher dimensions. Diffusion of a substance gives the same equation. Let $u(x, y, z, t)$ be the heat density and let $H(t)$ be the amount of heat contained in a region D. Then

$$H(t) = \iiint_D u \, dx\, dy\, dz.$$

The rate of change of the total heat in D is

$$\frac{dH}{dt} = \iiint_D u_t \, dx\, dy\, dz.$$

By Fourier's law of heat conduction, the heat flux through ∂D is proportional to the negative gradient of the temperature, i.e., $F = -k\nabla u$. Since heat is conserved, the total heat in D can only change by flowing through the boundary. By the divergence theorem

$$\iiint_D u_t \, dx\, dy\, dz = \frac{dH}{dt} = \iint_{\partial D} k(\nabla u \cdot \mathbf{n})\, dS = \iiint_D \operatorname{div}(k\nabla u)\, dx\, dy\, dz.$$

Therefore

$$\iiint_D (u_t - \operatorname{div}(k\nabla u))\, dx\, dy\, dz$$

for all regions D. It follows that the integrand must be zero, i.e.,

$$u_t - \operatorname{div}(k\nabla u) = 0.$$

Example 1.7 (Wave equation in higher dimensions)**.** Consider in 2 dimensions a vibrating drumhead with density ρ. Let D be a small set, say a circle or square, and let $|D|$ be the area of D. Newton's law applied to D states that

$$\rho|D|u_{tt} \approx ma = F = \kappa \int_D \nabla u \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the unit outward normal vector to the boundary ∂D, and u_{tt} is evaluated at any point in D. Using the divergence theorem we have

$$\frac{\rho}{\kappa}u_{tt} \approx \frac{1}{|D|}\iint_D \Delta u \, dx\, dy.$$

The quantity on the right is the average of Δu over D. If we take D to be a ball of radius $r > 0$ centered at x, then as $r \to 0$, the right hand side converges to $\Delta u(x)$, and we find that

$$u_{tt} - c^2 \Delta u = 0,$$

where $c = \sqrt{\kappa/\rho}$. The wave equation above also describes waves in higher dimensions (i.e., $n \geq 3$), such as sound or electromagnetic waves.

1.2 A brief list of other important PDE

The few PDE derived here are by no means all the important PDE. Here is a brief, and by no means complete, list of other important PDE.

- The inviscid Burgers equation
$$u_t + uu_x = 0$$
is a simplified one dimensional version of the Navier-Stokes equations that govern the motion of fluids. The viscous version of Burgers equation is
$$u_t + uu_x = \nu u_{xx}$$
where $\nu > 0$ is the viscosity.

- Fisher's equation
$$u_t - \Delta u = f(u)$$
with $f(u) = u(1-u)$ models population dynamics. Notice Fisher's equation is a diffusion equation with a forcing term $f(u)$ that depends on u. This is also called a reaction diffusion equation.

- The porous medium equation
$$u_t - \Delta(u^m) = 0,$$
where $m > 0$ is a constant, models flows (or diffusion) in porous rock or compacted soil.

- The Korteweg-deVries (Kdv) equation:
$$u_t + uu_x + u_{xxx} = 0$$
is a third-order equation for modeling water waves.

- The mean curvature motion equation
$$u_t - \|\nabla u\| \mathrm{div}\left(\frac{\nabla u}{\|\nabla u\|}\right) = 0$$
arises in physical systems that involve surface tension, such as soap film/bubbles and biological cell membranes, and many other fields of pure and applied mathematics. Here $\|\nabla u\|$ denotes the norm of the gradient vector, i.e.,
$$\|\nabla u\|^2 = u_x^2 + u_y^2 + u_z^2.$$

- The eikonal equation
$$f\|\nabla u\| = 1$$
arises in wave propagation, as well as computer vision and image processing.

- The Hamilton-Jacobi equation
$$u_t + H(\nabla u, x) = 0$$
arises in physics and optimal control theory.

- The minimal surface equation
$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}\right) = 0$$
describes the shape of soap bubbles.

- The Monge-Ampère equation
$$u_{xx}u_{yy} - u_{xy}^2 = f$$
has applications in Riemannian geometry, as well as optimal mass transportation problems.

- The Navier-Stokes equations:
$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u}$$
subject to the incompressibility constraint $\mathrm{div}\,\mathbf{u} = 0$, model the motion of fluids. Here $\mathbf{u} \in \mathbb{R}^3$ denotes the velocity field of the fluid, p the pressure, and the parameter $\nu > 0$ is the viscosity. There are actually 4 coupled PDE here, since the equation above is in vector form.