Math 5587 – Lecture 11

Jeff Calder

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1 Gibb's Phenomenon

The Fourier series for the function

$$f(x) = \begin{cases} -\frac{1}{2}, & \text{if } -\pi < x < 0 \\ +\frac{1}{2}, & \text{if } 0 < x < \pi \end{cases}$$

is

$$f(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{2}{\pi n} \sin(nx).$$

The Fourier series partial sums

$$S_N(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{N} \frac{2}{\pi n} \sin(nx), \tag{1}$$

converge pointwise to f provided we set $f(0) = 0$. We will show that the convergence here is *not* uniform, and furthermore, the partial sums S_N consistently overshoot the unit 1 jump by about 9% in the limit as $N \to \infty$. See Figure 1 for an illustration of the overshoot, which is called *Gibb's Phenomenon*.

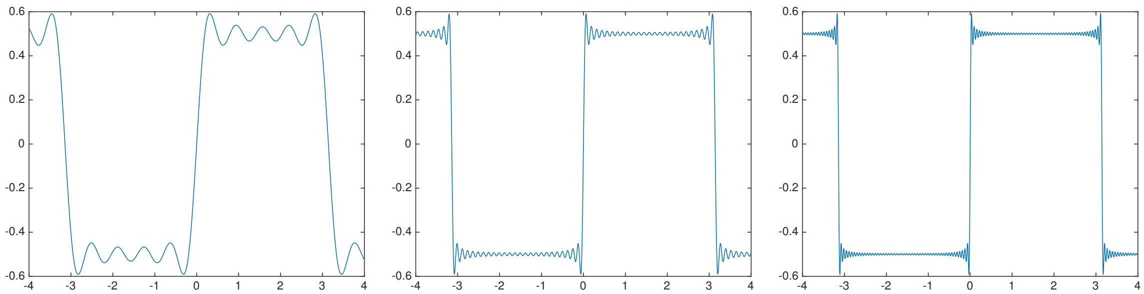
We will argue that

$$\lim_{N\to\infty} \max S_N = \frac{1}{\pi}\int_0^{\pi} \frac{\sin(\theta)}{\theta}\, d\theta \approx 0.59. \tag{2}$$

by viewing the sum in (1) as a Riemann sum for an integral similar to the one above. For a completely different proof, refer to Strauss 5.5.

To establish (2), we first consider the question of where the maximum of S_N is attained. If N is even, then $S_{N-1} = S_N$, and we can replace N by the odd number $N-1$. Therefore we may assume N is odd. We note that the highest frequency sine wave in the partial sum S_N is $\sin(Nx)$, and this function has a maximum at $Nx = \pi/2$. Thus, it is reasonable to suspect that the maximum of S_N is attained at some point x_N^* of the form $x_N^* = x/N$. Let's plug this into S_N and see what we get:

$$S_N\left(\frac{x}{N}\right) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{N} \frac{2}{\pi n} \sin\left(\frac{n}{N}x\right).$$



(a) $N = 10$ (b) $N = 50$ (c) $N = 100$

Figure 1: Depiction of Gibb's Phenomenon for a unit step function.

Let us rewrite this in a slightly different form:

$$S_N\left(\frac{x}{N}\right) = \frac{1}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{N}\left(\frac{n}{N}\right)^{-1}\sin\left(\frac{n}{N}x\right)\frac{2}{N}.$$

Let us set $\Delta\theta = 2/N$ and $\theta_n = n/N$. Then we have

$$S_N\left(\frac{x}{N}\right) = \frac{1}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{N}\frac{\sin(\theta_n x)}{\theta_n}\Delta\theta.$$

Since the sum is over all *odd* n, we have that $\Delta\theta = 2/N = \theta_{n+2} - \theta_n$ is the difference of two subsequent values of θ_n. Furthermore, $\theta_1 = 1/N$ and $\theta_N = 1$. Therefore, this is exactly a Riemann sum for the integral

$$\frac{1}{\pi}\int_0^1 \frac{\sin(\theta x)}{\theta}\,d\theta.$$

Therefore

$$\lim_{N\to\infty} S_N\left(\frac{x}{N}\right) = \frac{1}{\pi}\int_0^1 \frac{\sin(\theta x)}{\theta}\,d\theta = \frac{1}{\pi}\int_0^x \frac{\sin\theta}{\theta}\,d\theta =: \frac{1}{\pi}\mathrm{Si}(x).$$

The function $\mathrm{Si}(x)$ is called the *Sine Integral*. Like the error function, there is no closed form expression for $\mathrm{Si}(x)$. The integrand of the Sine Integral is often called the *cardinal sine function* or the *sinc* function and denoted

$$\operatorname{sinc}\theta := \frac{\sin\theta}{\theta}.$$

See Figure 2 for a plot of the sinc function.

We claim the maximum value of $\mathrm{Si}(x)$ occurs at $x = \pi$. To see this, we first note that $\operatorname{sinc}\theta$ has the same sign as $\sin\theta$ for $\theta > 0$. So for $n\pi < \theta < (n+1)\pi$, $\operatorname{sinc}\theta > 0$ for n even, and $\operatorname{sinc}\theta < 0$ for n odd. This tells us that $\mathrm{Si}(x)$ has local maximums at $x = n\pi$ for odd n. Since $\theta \mapsto 1/\theta$ is decreasing, and $|\sin\theta|$ is π-periodic, we have

$$\int_{n\pi}^{(n+1)\pi} |\operatorname{sinc}\theta|\,d\theta \geq \int_{(n+1)\pi}^{(n+2)\pi} |\operatorname{sinc}\theta|\,d\theta$$

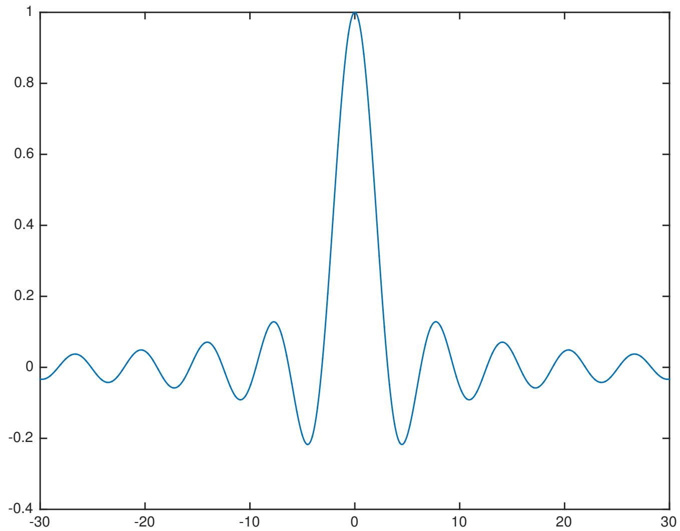


Figure 2: Plot of the function $\operatorname{sinc}\theta = \frac{\sin\theta}{\theta}$.

Therefore, for odd $n \in \mathbb{N}$ we have

$$\begin{aligned}
\operatorname{Si}((n+2)\pi) &= \int_0^{(n+2)n\pi} \operatorname{sinc}\theta \, d\theta \\
&= \operatorname{Si}(n\pi) + \int_{n\pi}^{(n+1)\pi} \operatorname{sinc}\theta \, d\theta + \int_{(n+1)\pi}^{(n+2)\pi} \operatorname{sinc}\theta \, d\theta \\
&= \operatorname{Si}(n\pi) - \int_{n\pi}^{(n+1)\pi} |\operatorname{sinc}\theta| \, d\theta + \int_{(n+1)\pi}^{(n+2)\pi} |\operatorname{sinc}\theta| \, d\theta \\
&\leq \operatorname{Si}(n\pi).
\end{aligned}$$

By induction, $\operatorname{Si}(n\pi) \leq \operatorname{Si}(\pi)$ for all odd $n \in \mathbb{N}$. This establishes the claim that the maximum of Si occurs at $x = \pi$. As an exercise, you may also wish to verify that $x = n\pi$ with n odd is a local maximum of Si by computing $\operatorname{Si}'(x) = 0$ and checking the sign of $\operatorname{Si}''(x)$.

Therefore, we have established that

$$\lim_{N\to\infty} S_N\left(\frac{\pi}{N}\right) = \frac{1}{\pi}\operatorname{Si}(\pi) = \frac{1}{\pi}\int_0^{\pi} \frac{\sin\theta}{\theta}\, d\theta \approx 0.58949,$$

where the value ≈ 0.58949 can be obtained by numerical integration. Since $f(0^+) = 0.5$, the overshoot is approximately 9% of the entire unit jump from $f(0^-) = -0.5$ to $f(0^+) = 0.5$.

To be entirely rigorous, we have actually only shown that

$$\liminf_{N\to\infty} \max S_N \geq \frac{1}{\pi}\operatorname{Si}(\pi) \approx 0.58949,$$

which shows that the overshoot is at least 9% in the limit as $N \to \infty$. To show that the limit exists, so that

$$\lim_{N\to\infty} \max S_N = \frac{1}{\pi}\operatorname{Si}(\pi),$$

takes a bit more work (but not too much). We can differentiate S_N to find that

$$S_N'(x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{N} \frac{2}{\pi} \cos(nx) = \frac{2}{\pi} \sum_{n=0}^{(N-1)/2} \cos((2n+1)x),$$

assuming N is odd. Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ we can derive a simpler expression for this sum in a similar way to our derivation of Dirichlet's kernel K_N in class. We have

$$S_N'(x) = \frac{\sin(x(N+1))}{\pi \sin x}.$$

As an exercise, you should fill in the details above. From this we can show that S_N attains its maximum at $x = \pi/(N+1)$, and using the argument above yields

$$\lim_{N\to\infty} S_N\left(\frac{\pi}{N+1}\right) = \frac{1}{\pi}\mathrm{Si}(\pi).$$