Math 5587 – Lecture 15

Jeff Calder

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Notation: We will work in $\mathbb{R}^3$, though all results hold with similar proof for arbitrary dimension. We write $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ for a point in $\mathbb{R}^3$.

1 The weak maximum principle

1.1 Topology

Let us first recall some basic Euclidean topology. For $\mathbf{x}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ and $r > 0$ we define the open ball of radius $r > 0$ centered at $\mathbf{x}_0$ to be

$$B(\mathbf{x}_0, r) = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x} - \mathbf{x}_0\| < r\}.$$

Recall that $\|x\| = \sqrt{x^2 + y^2 + z^2}$ is the Euclidean norm, or length, of the vector $\mathbf{x}$.

Definition 1 (Open set)**.** We say a set $D \subset \mathbb{R}^3$ is *open* if for every $\mathbf{x}_0 \in D$, there exists a radius $r > 0$ such that $B(\mathbf{x}_0, r) \subset D$.

In other words, a set D is open if every point $\mathbf{x}_0 \in D$ can be moved by a tiny amount in any direction and still remain in D. The open ball $B(\mathbf{x}_0, r)$ is open.

Definition 2 (Closure)**.** Given a set $D \subset \mathbb{R}^3$, the *closure* of D, denoted $\overline{D}$ is defined by

$$\overline{D} = \{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \lim_{n\to\infty} \mathbf{x}_n \text{ for some sequence } \mathbf{x}_n \in D\}.$$

We say a set D is *closed* if $D = \overline{D}$.

Thus, the closure of a set D is the set of all points $\mathbf{x} \in \mathbb{R}^3$ that can be reached by limits of sequences in D. By taking the constant sequence $\mathbf{x}_n = \mathbf{x} \in D$, we see that $D \subset \overline{D}$. For the open ball $B(\mathbf{x}_0, r)$ the closure is the closed ball

$$\overline{B(\mathbf{x}_0, r)} = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x} - \mathbf{x}_0\| \le r\}.$$

If D is open, then we often say that D is the *interior* of $\overline{D}$.

Definition 3 (Boundary)**.** Given an open set $D \subset \mathbb{R}^3$, the *boundary* of D, denoted ∂D, is defined by

$$\partial D = \overline{D} \setminus D := \{\mathbf{x} \in \overline{D} \mid \mathbf{x} \notin D\}.$$

For the ball $B(\mathbf{x}_0, r)$, the boundary is

$$\partial B(\mathbf{x}_0, r) = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x} - \mathbf{x}_0\| = r\}.$$

Definition 4 (Bounded set)**.** We say a set $D \subset \mathbb{R}^3$ is *bounded* if there exists $R > 0$ such that $D \subset B(\mathbf{0}, R)$, where $\mathbf{0} = (0, 0, 0)$.

In other words, D is bounded if there exists $R > 0$ such that for every $\mathbf{x} \in D$ we have $\|\mathbf{x}\| \leq R$. Every ball $B(\mathbf{x}_0, r)$ is bounded.

We note these are just one choice of definitions and others are available. For example, some books define closed sets to be the complements of open sets, and the closure is defined as the intersection of all closed sets containing D. A basic exercise in any course on real analysis is to verify that all the definitions are equivalent.

1.2 Poisson's equation

Let $D \subset \mathbb{R}^3$ be an open and bounded set, and consider Poisson's equation

$$-\Delta u = f \quad \text{in } D, \tag{1}$$

where Δu is the Laplacian defined by

$$\Delta u := u_{xx} + u_{yy} + u_{zz}.$$

We will consider solutions u in the space $C^2(\overline{D})$. For general k, $C^k(\overline{D})$ is the collection of functions $u : \overline{D} \to \mathbb{R}$ that are k-times continuously differentiable on $\overline{D}$. We consider $C^2(\overline{D})$ because Δu involves second derivatives of u, and so these should be defined and continuous for the equation to make sense classically.

We give solutions of Laplace's equation $\Delta u = 0$ a special name.

Definition 5. We say $u \in C^2(\overline{D})$ is *harmonic* in D if $\Delta u = 0$ in D.

Recall that Poisson's equation (1) is steady state for the heat equation

$$u_t - \Delta u = f$$

and the wave equation

$$u_{tt} - \Delta u = f.$$

For the heat equation, $f(\mathbf{x})$ represents the rate per unit volume that heat is being added or removed at position $\mathbf{x}$. For the wave equation $f(\mathbf{x})$ represents the magnitude of an external force applied in the vertical direction at position $\mathbf{x}$.

Let's consider steady state for the heat equation when $f = 0$ for a moment. This means $u_t = 0$ so the heat density is constant in time. Since $f = 0$, there is no heat being added or removed from the system in the interior D. We claim from a formal standpoint that u cannot have an interior max or min, i.e., the max and min must be attained on the boundary ∂D. To see why, recall that the heat equation $u_t - \Delta u = 0$ decreases interior maximums and increases interior minimums. If u had a strict maximum inside D, for instance, then the heat equation would decrease the value of this max, and so u could not be a steady state solution of the heat equation.

The argument above is purely formal, but it suggests that harmonic functions satisfy some type of maximum principle. Formalizing the maximum principle is an important step in our study of Poisson's equation.

Theorem 1 (Weak maximum principle)**.** *Let $D \subset \mathbb{R}^3$ be an open and bounded set, and suppose $u \in C^2(\overline{D})$ satisfies*

$$\Delta u \geq 0 \quad \text{in } D. \tag{2}$$

Then

$$\max_{\mathbf{x} \in \overline{D}} u(\mathbf{x}) = \max_{\mathbf{x} \in \partial D} u(\mathbf{x}). \tag{3}$$

The maximum principle states that whenever $\Delta u \geq 0$ in D, u must attain its maximum value over $\overline{D}$ on the boundary ∂D. It does not preclude u from also attaining its maximum inside D. We will prove later the *Strong Maximum Principle*, which says that the only time u can attain its max inside D is when u is constant.

The maximum principle relies on the following lemma.

Lemma 1 (Necessary conditions for maxima)**.** *Let $D \subset \mathbb{R}^3$ be open, and suppose there exists $\mathbf{x}_0 \in D$ such that*

$$u(\mathbf{x}) \leq u(\mathbf{x}_0) \quad \text{for all } \mathbf{x} \in D.$$

Then

$$\Delta u(\mathbf{x}_0) \leq 0. \tag{4}$$

Proof. Since D is open, there exists $r > 0$ such that $B(\mathbf{x}_0, r) \subset D$. Therefore $x \mapsto u(x, y_0, z_0)$ has a maximum at x_0 over the interval $x \in (x_0 - r, x_0 + r)$. By the second derivative test $u_{xx}(x_0, y_0, z_0) \leq 0$. Similar arguments show that $u_{yy}(\mathbf{x}_0) \leq 0$ and $u_{zz}(\mathbf{x}_0) \leq 0$. Therefore

$$\Delta u(\mathbf{x}_0) = u_{xx}(\mathbf{x}_0) + u_{yy}(\mathbf{x}_0) + u_{zz}(\mathbf{x}_0) \leq 0. \qquad \square$$

We now have the proof of the maximum principle Theorem 1.

Proof. Let $\varepsilon > 0$ and define

$$w(\mathbf{x}) = u(\mathbf{x}) + \varepsilon x^2,$$

where $\mathbf{x} = (x, y, z) \in \overline{D}$. Since D is bounded, there exists $R > 0$ such that $D \subset B(\mathbf{0}, R)$, and therefore $x^2 \leq R^2$ for all $\mathbf{x} \in \overline{D}$. It follows that

$$u(\mathbf{x}) \leq w(\mathbf{x}) \leq u(\mathbf{x}) + \varepsilon R^2 \quad \text{for all } \mathbf{x} \in \overline{D}. \tag{5}$$

We also compute that

$$\Delta w(\mathbf{x}) = \Delta u(\mathbf{x}) + 2\varepsilon \geq 2\varepsilon > 0 \quad \text{for all } \mathbf{x} \in D. \tag{6}$$

Since w is a continuous function on a closed and bounded set $\overline{D}$, w attains its maximum over $\overline{D}$ at some point $\mathbf{x}_0 \in \overline{D}$. By (6) and Lemma 1 we know $\mathbf{x}_0 \notin D$; therefore $\mathbf{x}_0 \in \partial D$ and we have

$$\begin{aligned} \max_{\mathbf{x} \in \overline{D}} u(\mathbf{x}) \leq \max_{\mathbf{x} \in \overline{D}} w(\mathbf{x}) &= \max_{\mathbf{x} \in \partial D} w(\mathbf{x}) \\ (5) &\leq \max_{\mathbf{x} \in \partial D} u(\mathbf{x}) + \varepsilon R^2. \end{aligned}$$

Sending $\varepsilon \to 0^+$ we have

$$\max_{\mathbf{x} \in \overline{D}} u(\mathbf{x}) \leq \max_{\mathbf{x} \in \partial D} u(\mathbf{x}).$$

Since $\partial D \subset \overline{D}$, the opposite inequality $\max_{\mathbf{x}\in\partial D} u(\mathbf{x}) \leq \max_{\mathbf{x}\in\overline{D}} u(\mathbf{x})$ is true trivially. Therefore

$$\max_{\mathbf{x}\in\overline{D}} u(\mathbf{x}) = \max_{\mathbf{x}\in\partial D} u(\mathbf{x}). \qquad \square$$

We also have the corresponding minimum principle.

Corollary 1 (Weak minimum principle)**.** *Let $D \subset \mathbb{R}^3$ be an open and bounded set, and suppose $u \in C^2(\overline{D})$ satisfies*

$$\Delta u \leq 0 \quad \textit{in } D. \tag{7}$$

Then

$$\min_{\mathbf{x}\in\overline{D}} u(\mathbf{x}) = \min_{\mathbf{x}\in\partial D} u(\mathbf{x}). \tag{8}$$

Proof. Apply the weak maximum principle to $v = -u$. $\square$

We can also restate the maximum and minimum principles as in the following corollary, whose proof is immediate.

Corollary 2. *Let $D \subset \mathbb{R}^3$ be open and bounded, and suppose u is harmonic in D. Then*

$$m \leq u(\mathbf{x}) \leq M \quad \textit{for all } \mathbf{x} \in \overline{D}, \tag{9}$$

where $m = \min_{\mathbf{x}\in\partial D} u(\mathbf{x})$ and $M = \max_{\partial D} u(\mathbf{x})$.

The maximum principle is a very powerful tool for proving uniqueness and stability (among many other properties) of Poisson's equation.

Lemma 2. *Let $D \subset \mathbb{R}^3$ be open and bounded, and let $u \in C^2(\overline{D})$ be a solution of*

$$\left.\begin{aligned} -\Delta u &= f \quad \textit{in } D \\ u &= g \quad \textit{on } \partial D. \end{aligned}\right\} \tag{10}$$

Then there exists $R > 0$ depending only on D such that

$$\max_{\mathbf{x}\in\overline{D}} |u(\mathbf{x})| \leq \max_{\mathbf{x}\in\partial D} |g(\mathbf{x})| + \frac{R^2}{6} \max_{\mathbf{x}\in\overline{D}} |f(\mathbf{x})|. \tag{11}$$

Proof. Since D is bounded, there exists $R > 0$ such that $D \subset B(\mathbf{0}, R)$. Let $A = \max_{\mathbf{x}\in\partial D} |g(\mathbf{x})|$, $B = \max_{\mathbf{x}\in\overline{D}} |f(\mathbf{x})|$, and define

$$v(\mathbf{x}) = u(\mathbf{x}) + \frac{B}{6}(x^2 + y^2 + z^2 - R^2) - A,$$

where $\mathbf{x} = (x, y, z)$. By the choice of R, $x^2 + y^2 + z^2 \leq R^2$ for all R, therefore for $\mathbf{x} \in \partial D$

$$v(\mathbf{x}) \leq u(\mathbf{x}) - A = g(\mathbf{x}) - \max_{\mathbf{x}\in\partial D} |g(\mathbf{x})| \leq 0.$$

We also compute

$$\Delta v(\mathbf{x}) = \Delta u(\mathbf{x}) + B = -f(\mathbf{x}) + \max_{\mathbf{x}\in\overline{D}} |f(\mathbf{x})| \geq 0.$$

By the maximum principle Theorem 1 we have $v(\mathbf{x}) \le 0$ for all $\mathbf{x} \in \overline{D}$. Therefore

$$u(\mathbf{x}) \le A + \frac{B}{6}(R^2 - x^2 - y^2 - z^2) \le \max_{\mathbf{x}\in\partial D} |g(\mathbf{x})| + \frac{R^2}{6} \max_{\mathbf{x}\in\overline{D}} |f(\mathbf{x})|,$$

for all $x \in \overline{D}$. Applying the same argument to $-u$ yields

$$-u(\mathbf{x}) \le \max_{\mathbf{x}\in\partial D} |g(\mathbf{x})| + \frac{R^2}{6} \max_{\mathbf{x}\in\overline{D}} |f(\mathbf{x})|.$$

Therefore

$$|u(\mathbf{x})| \le \max_{\mathbf{x}\in\partial D} |g(\mathbf{x})| + \frac{R^2}{6} \max_{\mathbf{x}\in\overline{D}} |f(\mathbf{x})|$$

for all $\mathbf{x} \in \overline{D}$, which completes the proof. $\square$

Let us give an application of Lemma 2 to uniqueness and stability of the Dirichlet problem for Poisson's equation. Suppose $u, v \in C^2(\overline{D})$ are solutions of

$$\left.\begin{aligned} -\Delta u &= f_1 \quad \text{in } D \\ u &= g_1 \quad \text{on } \partial D, \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} -\Delta v &= f_2 \quad \text{in } D \\ v &= g_2 \quad \text{on } \partial D, \end{aligned}\right\} \tag{12}$$

respectively, where $D \subset \mathbb{R}^3$ is open and bounded. Then $w := u - v$ is a solution of

$$\left.\begin{aligned} -\Delta w &= f_1 - f_2 \quad \text{in } D \\ w &= g_1 - g_2 \quad \text{on } \partial D, \end{aligned}\right\} \tag{13}$$

as the equation is linear. Applying Lemma 2 to w we have

$$\max_{\mathbf{x}\in\overline{D}} |u_1(\mathbf{x}) - u_2(\mathbf{x})| \le \max_{\mathbf{x}\in\partial D} |g_1(\mathbf{x}) - g_2(\mathbf{x})| + \frac{R^2}{6} \max_{\mathbf{x}\in\overline{D}} |f_1(\mathbf{x}) - f_2(\mathbf{x})|. \tag{14}$$

Eq. (14) gives both uniqueness and stability. Indeed, if $f_1 = f_2$ and $g_1 = g_2$, then $u_1 = u_2$, which is uniqueness. Furthermore, if f_1 is close to f_2, and g_1 is close to g_2, then u_1 is similarly close to u_2. This is a stability result; small changes in the data yield small changes in the solution.

2 Energy methods

Before discussing energy methods, let us recall the divergence theorem and Green's identities.

2.1 Green's identities

Recall for a vector field $\mathbf{v}(\mathbf{x}) = (\mathbf{v}_1(\mathbf{x}), \mathbf{v}_2(\mathbf{x}), \mathbf{v}_3(\mathbf{x}))$, the divergence of $\mathbf{v}$ is defined as

$$\operatorname{div}(\mathbf{v}) = \frac{\partial \mathbf{v}_1}{\partial x} + \frac{\partial \mathbf{v}_2}{\partial y} + \frac{\partial \mathbf{v}_3}{\partial z}.$$

If $\mathbf{v} = \nabla u = (u_x, u_y, u_z)$ for a function $u(x, y, z)$ then

$$\operatorname{div}(\nabla u) = u_{xx} + u_{yy} + u_{zz} = \Delta u.$$

We will assume our bounded open set $D \subset \mathbb{R}^3$ has a smooth boundary ∂D so that we can define at each point $\mathbf{x} \in \partial D$ the unit outward normal vector $\mathbf{n} = \mathbf{n}(\mathbf{x})$. Let us recall the divergence theorem

$$\iiint_D \operatorname{div}(\mathbf{v})\, d\mathbf{x} = \iint_{\partial D} \mathbf{v} \cdot \mathbf{n}\, dS,$$

where $d\mathbf{x} = dxdydz$. By making special choices for $\mathbf{v}$, we can deduce important integration identities collectively called Green's identities.

First, we let $\mathbf{v} = \operatorname{div}(v\nabla u)$ for functions $u, v \in C^2(\overline{D})$. Using the product rule we find

$$\mathbf{v} = \operatorname{div}(v\nabla u) = \nabla u \cdot \nabla v + v\Delta u.$$

Plugging this into the divergence theorem yields *Green's first identity*

$$\boxed{\iint_{\partial D} v\frac{\partial u}{\partial \mathbf{n}}\, dS = \iiint_D \nabla u \cdot \nabla v\, d\mathbf{x} + \iiint_D v\Delta u\, d\mathbf{x},}$$

where $\frac{\partial u}{\partial \mathbf{n}} = \nabla u \cdot \mathbf{n}$ is the normal derivative of u.

Swapping the roles of u and v in Green's first identity we have

$$\iint_{\partial D} u\frac{\partial v}{\partial \mathbf{n}}\, dS = \iiint_D \nabla u \cdot \nabla v\, d\mathbf{x} + \iiint_D u\Delta v\, d\mathbf{x}.$$

Subtracting this from Green's first identity we have *Green's second identity*

$$\boxed{\iiint_D (u\Delta v - v\Delta u)\, d\mathbf{x} = \iint_{\partial D} \left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}}\right) dS.}$$

A special case of Green's second identity is obtained by taking $v = 1$ to find that

$$\boxed{\iiint_D \Delta u\, d\mathbf{x} = \iint_{\partial D} \frac{\partial u}{\partial \mathbf{n}}\, dS.}$$

Green's identities are extensions of the familiar one-dimensional integration by parts formula to higher dimensions. All of the identities above hold for arbitrary dimension (not just 3), and are commonly referred to as just *integration by parts* formulas.

2.2 The Dirichlet problem

We consider again the Dirichlet problem for Poisson's equation

$$\left.\begin{aligned} -\Delta u &= f \quad \text{in } D \\ u &= g \quad \text{on } \partial D. \end{aligned}\right\} \tag{15}$$

Energy methods give us a quick and easy proof of uniqueness. Indeed, suppose $u, v \in C^2(\overline{D})$ are solutions of (15). Then $w := u - v$ satisfies $\Delta w = 0$ in D and $w = 0$ on ∂D. Applying Green's first identity to w (use $u = w$ and $v = u$ in the identity) yields

$$0 = \iiint_D \nabla w \cdot \nabla w\, d\mathbf{x} = \iiint_D \|\nabla w\|^2\, d\mathbf{x}.$$

The integrand $\|\nabla w\|^2$ is nonnegative, and it's integral is zero. Since ∇w is continuous, we must have $\nabla w = 0$ in D. Since $w = 0$ on ∂D, w is constant and $w = 0$ throughout D. Hence $u = v$, so we have uniqueness.

Proving stability of (15) with energy methods is a bit trickier, and requires a Poincaré inequality (similar to HW6 Problem 7, but in $\mathbb{R}^3$). However, for a similar equation, energy methods can be used to prove stability with little difficulty.

Exercise 1. Let $u \in C^2(\overline{D})$ be a solution of

$$\left.\begin{aligned} u - \Delta u &= f \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D. \end{aligned}\right\} \tag{16}$$

Use energy methods to show that

$$\iiint_D u^2 + \|\nabla u\|^2 \, d\mathbf{x} \leq \iiint_D f^2 \, d\mathbf{x}. \tag{17}$$

Indicate how this is a stability estimate for (16). [Hint: Multiple the PDE by u, integrate both sides over D, and then use Green's first identity. Use Cauchy's inequality $2ab \leq a^2 + b^2$ on the right hand side.]

2.3 The Neumann problem

We now consider the Neumann problem for Poisson's equation

$$\left.\begin{aligned} -\Delta u &= f \quad \text{in } D \\ \frac{\partial u}{\partial \mathbf{n}} &= g \quad \text{on } \partial D. \end{aligned}\right\} \tag{18}$$

We first note that if u is a solution of (18), then so is $v = u + C$ for any constant C. So we do not expect to get uniqueness, but we can hope to show that any two solutions must differ by a constant. For this, we also assume the domain D is *connected*. This means that between any two points $\mathbf{x}_1$ and $\mathbf{x}_2$ in D there exists a continuous path contained entirely in D starting at x_1 and ending at $\mathbf{x}_2$.

As usual, let u and v be two solutions of (18) and define $w := u - v$. Then w satisfies $\Delta w = 0$ in D and $\frac{\partial w}{\partial \mathbf{n}} = 0$ on ∂D. Using Green's identity we again find that

$$\iiint_D \|\nabla w\|^2 \, d\mathbf{x} = 0.$$

Therefore $\nabla w = 0$ throughout D. Since D is connected, $w = C$ for some constant C.[1] Therefore $u = v + C$. This is the best we can prove, given the discussion above. Therefore the solution of (18) is unique up to a constant.

Energy methods can also give us information about existence of solutions. Consider now the homogeneous Neumann problem

$$\left.\begin{aligned} -\Delta u &= f \quad \text{in } D \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial D. \end{aligned}\right\} \tag{19}$$

[1] Since $\nabla w = 0$, we can show that w is constant along any path in D, and since D is connected, w is constant on D. If D was not connected, say it was the union of two disjoint balls, then w could assume a different value on each ball, and would instead be piecewise constant.

Suppose a solution $u \in C^2(\overline{D})$ of (19) exists. Then using the special case immediately after Green's second identity we have

$$\iiint_D f\, d\mathbf{x} = -\iiint_D \Delta u\, d\mathbf{x} = -\iint_{\partial D} \frac{\partial u}{\partial \mathbf{n}}\, dS = 0.$$

Therefore

$$\boxed{\iiint_D f\, d\mathbf{x} = 0} \tag{20}$$

is a necessary condition for the existence of a solution u of (19). That is, if $\iiint_D f\, d\mathbf{x} \neq 0$, then no solution exists.

At first, the necessary condition (20) may seem peculiar. It is, in fact, very natural when one considers that (19) is steady state for the heat equation

$$\left.\begin{aligned} u_t - \Delta u &= f \quad \text{in } D \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial D. \end{aligned}\right\} \tag{21}$$

subject to some initial condition $u(\mathbf{x}, 0) = g(\mathbf{x})$. Let $H(t)$ be the total heat in the body D, which is given by

$$H(t) = \iiint_D u(\mathbf{x}, t)\, d\mathbf{x}.$$

The rate of change of H in t is

$$H'(t) = \iiint_D u_t(\mathbf{x}, t)\, d\mathbf{x} = \iiint_D \Delta u + f\, d\mathbf{x} = \iiint_D \Delta u\, d\mathbf{x} + \iiint_D f\, d\mathbf{x}.$$

Using Green's identity on the first term on the right yields

$$H'(t) = \iint_{\partial D} \frac{\partial u}{\partial \mathbf{n}}\, dS + \iiint_D f\, d\mathbf{x} = \iiint_D f\, d\mathbf{x},$$

since $\frac{\partial u}{\partial \mathbf{n}} = 0$. If we have any hope of the heat equation reaching steady state, then we must have $H'(t) = 0$, that is, the total heat is conserved. In this light, (20) is a completely natural necessary condition for the existence of a solution of the steady state equation (19).

Furthermore, the steady state interpretation of Poisson's equation sheds light on why solutions of (19) are unique only up to a constant. Indeed, whenever (20) holds we have

$$H(t) = \iint_D u(\mathbf{x}, t)\, d\mathbf{x} = \iiint_D g(\mathbf{x})\, d\mathbf{x},$$

where $u(\mathbf{x}, 0) = g(\mathbf{x})$. We can of course choose initial conditions $g(\mathbf{x})$ with any total heat $H = \iiint_D g(\mathbf{x})\, d\mathbf{x}$ we wish. Each choice of H yields a different steady state solution (with different total heat), and the analysis above shows that all choices of H simply yield vertical translations $u + C$ of one solution u of (19).