# Math 5587 – Lecture 5

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## 1 Energy methods for the heat equation

Consider a thin insulted rod of length $l > 0$ and thermal conductivity $k > 0$. Assume the ends of the rod are insulated. Then the temperature (or heat density) along the rod $u(x, t)$ satisfies the heat equation with Neumann boundary conditions

$$\begin{cases} u_t - ku_{xx} = 0, & \text{for } 0 < x < l \\ u_x(0, t) = u_x(l, t) = 0, & \text{for } t \geq 0. \end{cases} \tag{1}$$

As we did for the wave equation, we can use energy methods to prove uniqueness and stability of the heat equation. The obvious quantity to consider is the total heat

$$H(t) = \int_0^l u(x, t)\, dx.$$

We have

$$\frac{dH}{dt} = \int_0^l u_t(x, t)\, dx = k \int_0^l u_{xx}(x, t)\, dx = k(u_x(l, t) - u_x(0, t)) = 0.$$

due to the Neumann boundary conditions. Therefore, as expected the total heat is conserved. Notice this would not be the case with Dirichlet conditions $u(0, t) = u(l, t) = 0$, since in this case heat can escape or enter the rod through the ends.

Although it's a nice observation that heat is conserved, it unfortunately does not help us prove stability or uniqueness. We need to consider a different type of energy for this. To this end consider

$$I(t) = \frac{1}{2} \int_0^l u(x, t)^2\, dx.$$

The quantity $I(t)$ does not represent a physical quantity in the context of heat diffusion, but is useful nonetheless for studying the heat equation. Notice that

$$\frac{dI}{dt} = \int_0^l uu_t\, dx = k \int_0^l uu_{xx}\, dx = kuu_x\Big|_0^l - k \int_0^l u_x^2\, dx \leq 0,$$

since $u_x(0, t) = u_x(l, t) = 0$ and $u_x^2 \geq 0$. Therefore, $I$ may not be conserved, but we do know that $I$ is a decreasing function of $t$. The reader should notice that the same result holds with Dirichlet conditions $u(0, t) = u(l, t) = 0$.

Since $I$ is decreasing, we have $I(t) \leq I(0)$ for $t \geq 0$ and therefore

$$\int_0^l u(x,t)^2 \, dx \leq \int_0^l u(x,0)^2 \, dx \quad \text{for} \quad t \geq 0. \tag{2}$$

The identity above proves stability and uniqueness for the heat equation. Indeed, suppose we have two solution $u_1(x,t)$ and $u_2(x,t)$ of the heat equation (1). Then $w(x,t) := u_1(x,t) - u_2(x,t)$ is a solution of the same heat equation. Suppose we have the initial conditions

$$u_1(x,0) = f_1(x) \quad \text{and} \quad u_2(x,0) = f_2(x).$$

Applying the energy estimate (2) to $w$ we have

$$\int_0^l (u_1(x,t) - u_2(x,t))^2 \, dx \leq \int_0^l (f_1(x) - f_2(x))^2 \, dx \quad \text{for} \quad t \geq 0. \tag{3}$$

Hence, if $f_1 = f_2$ then $u_1 = u_2$, which proves uniqueness. Furthermore, if $f_1$ and $f_2$ are close in the square integrable sense as above, then $u_1$ and $u_2$ are similarly close in the square integrable sense for all time. This means that small perturbations of the initial conditions yield similarly small perturbations in the solutions, and so the heat equation is stable.

Here, we are measuring the closeness of two functions with the squared $L^2$ norm

$$\|u_1 - u_2\|_{L^2(0,l)}^2 := \int_0^l (u_1 - u_2)^2 \, dx.$$

Next we will show how to use the maximum principle to obtain stronger stability results for the heat equation.

## 2 The maximum principle

Let $u(x,t)$ be a solution of the heat equation

$$u_t - ku_{xx} = 0.$$

Suppose that $u$ has a maximum at a point $(x_0, t_0)$. Then $u_{xx}(x_0, t_0) \leq 0$, since the function $g(x) = u(x, t_0)$ has a maximum at $x_0$ and so $g''(x_0) = u_{xx}(x_0, t_0) \leq 0$. Therefore

$$u_t(x_0, t_0) = ku_{xx}(x_0, t_0) \leq 0.$$

This shows that the heat equation decreases maxima of $u$, and a corresponding argument shows that the heat equation increases minima. This is of course just another way of observing that heat flows from hot regions to cold regions, and not the other way around. Thus hot regions (maxima of $u$) get progressively colder ($u$ decreases), while cold regions (minima of $u$) get warmer ($u$ increases).

Although this is a simple observation, it is an extremely powerful tool for analyzing PDE. To formulate the maximum principle properly we need to introduce some notation. We define the rectangle

$$U_T = (a,b) \times (0,T] = \{(x,t) \, : \, a < x < b \text{ and } 0 < t \leq T\}.$$

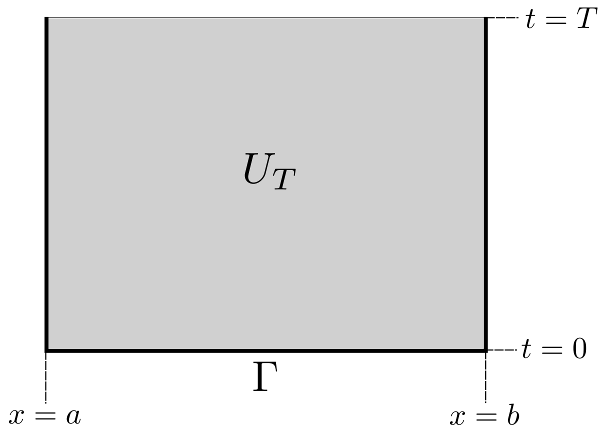


Figure 1: An illustration of $U_T$ and its parabolic boundary (i.e., the sides and base) $\Gamma$. The set $U_T$ is the shaded region, while $\Gamma$ is the bold portion of the boundary of $U_T$. The closure $\overline{U_T}$ consists of $U_T$ and all four of its sides.

We denote by $\overline{U_T}$ the closure of the rectangle, given by

$$\overline{U_T} = [a, b] \times [0, T] = \{(x, t) \, : \, a \leq x \leq b \text{ and } 0 \leq t \leq T\}.$$

Let $\Gamma \subset \overline{U_T}$ denote the sides and base of the rectangle, i.e.,

$$\Gamma = \{(x, t) \in \overline{U_T} \, : \, x = a, \; x = b, \; \text{ or } t = 0\}.$$

The set $\Gamma$ is called the *parabolic boundary* of $U_T$. See Figure 1 for a depicition of $U_T$ and $\Gamma$.

**Theorem 1** (Weak maximum principle)**.** *Suppose that $u(x, t)$ satisfies*

$$u_t - k u_{xx} \leq 0 \quad \textit{in} \quad U_T. \tag{4}$$

*Then*

$$\max_{\overline{U_T}} u = \max_{\Gamma} u. \tag{5}$$

A function $u$ satisfying (4) is called a *subsolution* of the heat equation. Similarly, a function $v(x, t)$ satisfying

$$v_t - k v_{xx} \geq 0$$

is called a *supersolution*. The weak maximum principle states that the maximum value of any subsolution of the heat equation on $\overline{U_T}$ is attained on the parabolic boundary $\Gamma$, that is, on the sides or base of the rectangle. The strong maximum principle (which we shall not prove here), states further that whenever the maximum is also attained inside $U_T$, the function $u$ must be constant.

Before giving the proof of Theorem 1, we have a short lemma.

**Lemma 1** (Necessary conditions for interior maxima)**.** *If $u$ attains its maximum over $\overline{U_T}$ at a point $(x_0, t_0) \in U_T$ then*

$$u_t(x_0, t_0) \geq 0 \quad \textit{and} \quad u_{xx}(x_0, t_0) \leq 0. \tag{6}$$

*In particular*

$$u_t(x_0, t_0) - k u_{xx}(x_0, t_0) \geq 0. \tag{7}$$

*Proof.* Let $h > 0$. Since $(x_0, t_0) \in U_T$ we have $t_0 > 0$. Therefore $(x_0, t_0 - h) \in U_T$ for small $h > 0$. It follows that $u(x_0, t_0) \geq u(x_0, t_0 - h)$ and thus

$$u_t(x_0, t_0) = \lim_{h \to 0^+} \frac{u(x_0, t_0) - u(x_0, t_0 - h)}{h} \geq 0.$$

This establishes the first part of (6).

For the second part, let $g(x) = u(x, t_0)$. Since $(x_0, t_0) \in U_T$, $a < x_0 < b$. Therefore $g$ has a maximum at $x_0$ and

$$u_{xx}(x_0, t_0) = g''(x_0) \leq 0. \qquad \square$$

We now have the proof of Theorem 1

*Proof of Theorem 1.* Suppose for a moment that

$$u_t - k u_{xx} < 0 \quad \text{in} \quad U_T. \tag{8}$$

Since this is incompatible with (7), $u$ cannot have a maximum in $U_T$. Since $u$ is continuous on the closed and bounded set $\overline{U_T}$, $u$ must attain its maximum at a point $(x_0, t_0) \in \Gamma$, and so

$$\max_{\overline{U_T}} u = \max_{\Gamma} u.$$

Now, $u$ only satisfies (4), so the above argument does not directly apply. The trick here is to modify $u$ slightly so that (8) is satisfied. That is, we need to turn a subsolution of the heat equation into a strict subsolution. With this in mind, let $\varepsilon > 0$ and define

$$v(x, t) := u(x, t) - \varepsilon t.$$

Then

$$v_t = u_t - \varepsilon \quad \text{and} \quad v_{xx} = u_{xx}.$$

Therefore

$$v_t - k v_{xx} = u_t - k u_{xx} - \varepsilon \leq -\varepsilon < 0.$$

Hence $v$ is a strict subsolution, that is, $v$ satisfies (8). By the argument above

$$\max_{\overline{U_T}} v = \max_{\Gamma} v \leq \max_{\Gamma} u.$$

The last step follows from the fact that $v \leq u$. Since $0 \leq t \leq T$, we have

$$u = v + \varepsilon t \leq v + \varepsilon T.$$

Therefore

$$\max_{\overline{U_T}} u \leq \max_{\overline{U_T}} v + \varepsilon T \leq \max_{\Gamma} u + \varepsilon T.$$

Since $\varepsilon > 0$ was an arbitrary real number, we can send $\varepsilon \to 0^+$ to find that

$$\max_{\overline{U_T}} u \leq \max_{\Gamma} u.$$

We of course have $\max_{\overline{U_T}} u \geq \max_{\Gamma} u$, since $\Gamma \subset \overline{U_T}$. This completes the proof. $\square$

Notice the main idea in the proof is to show that the subsolution property (4) is incompatible with the necessary conditions for an interior maxima (7). Any argument making use of the necessary conditions for a maximum is called a *maximum principle* argument. The maximum principle is a very widely applicable tool in the theory PDE, and applies to very general classes of nonlinear PDE as well. However, since necessary conditions for a maxima only give information about 1st and 2nd derivatives, maximum principle techniques are not useful for higher order equations.

We also have a corresponding minimum principle.

**Corollary 1** (Minimum principle)**.** *Suppose that $u(x,t)$ satisfies*

$$u_t - ku_{xx} \geq 0 \quad in \quad U_T. \tag{9}$$

*Then*

$$\min_{\overline{U_T}} u = \min_{\Gamma} u. \tag{10}$$

*Proof.* We simply apply Theorem 1 to $v = -u$. □

### 2.1 Stability and uniqueness via the maximum principle

The maximum principle can be used to prove stability and uniqueness. Consider the heat equation with homogeneous Dirichlet boundary conditions:

$$\begin{cases} u_t - ku_{xx} = 0 & \text{in } U_T \\ u(0,t) = u(l,t) = 0 & \text{for } t \geq 0. \end{cases} \tag{11}$$

Here, we set $a = 0$ and $b = l$ in the definitions of $U_T$ and $\Gamma$. Let $u_1$ and $u_2$ be two solutions of (11) with initial conditions

$$u_1(x,0) = f_1(x) \quad \text{and} \quad u_2(x,0) = f_2(x).$$

Let

$$C = \max_{0\leq x\leq l} |f_1(x) - f_2(x)|.$$

Then $w(x,t) := u_1(x,t) - u_2(x,t)$ is a solution of the heat equation with Dirichlet boundary conditions and initial condition

$$w(x,0) = f_1(x) - f_2(x).$$

Notice that $w(x,0) \leq C$. By the maximum principle (Theorem 1), $w$ attains its maximum over the rectangle on the parabolic boundary $\Gamma$. On the sides of $\Gamma$, $w(0,t) = w(l,t) = 0$. On the base, $w(x,0) \leq C$. Therefore

$$\max_{\overline{U_T}}(u_1 - u_2) = \max_{\overline{U_T}} w \leq C = \max_{0\leq x\leq l} |f_1(x) - f_2(x)|.$$

We can reverse the roles of $u_1$ and $u_2$ to find that

$$\max_{\overline{U_T}}(u_2 - u_1) \leq \max_{0\leq x\leq l} |f_1(x) - f_2(x)|.$$

It follows that

$$\max_{\overline{U_T}} |u_1 - u_2| \leq \max_{0 \leq x \leq l} |f_1(x) - f_2(x)|. \tag{12}$$

As before, this type of estimate gives us uniqueness and stability. Indeed, if $f_1 = f_2$, then the right hand side is zero, and hence $u_1 = u_2$ on $\overline{U_T}$. If $f_1$ and $f_2$ are close in a uniform sense (that is, the maximum absolute value of their difference is small), then the same is true for $u_1$ and $u_2$. Hence, small perturbations in the initial data yield correspondingly small perturbations in the solutions.

Here, we are measuring the distance between functions in the so-called $L^\infty$ norm

$$\|f_1 - f_2\|_{L^\infty(0,l)} := \max_{0 \leq x \leq l} |f_1(x) - f_2(x)|.$$

The $L^\infty$ norm is much stronger than the $L^2$ norm. By this, we mean that functions that are close in the $L^\infty$ sense are also close in the $L^2$ sense. This is expressed by the inequality

$$\|f_1 - f_2\|^2_{L^2(0,l)} = \int_0^l (f_1(x) - f_2(x))^2 \, dx \leq l \|f_1 - f_2\|^2_{L^\infty(0,l)}.$$

The opposite is not true. That is, there exist functions that are close in the $L^2$ sense, but are very far apart in the $L^\infty$ norm. For example, consider $f_1(x) = 0$ and

$$f_2(x) = \begin{cases} n, & \text{if } 0 \leq x \leq \frac{1}{n^3} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|f_1 - f_2\|^2_{L^2(0,l)} = \int_0^{\frac{1}{n^3}} n^2 \, dx = \frac{1}{n},$$

and

$$\|f_1 - f_2\|_{L^\infty(0,l)} = n.$$

When $n$ is large, $f_1$ and $f_2$ are very close in the $L^2$ norm, yet very far apart in the $L^\infty$ norm. This is not a paradox–it is simply due to the fact that the norms are capturing different discrepancies between the two functions. The $L^2$ norm measures the average squared distance, while the $L^\infty$ norm measures the maximum absolute distance.

There are other various $L^p$ norms that are often used in PDE and analysis for measuring the distance between functions. We have

$$\|f_1 - f_2\|^p_{L^p(0,l)} := \int_0^l |f_1(x) - f_2(x)|^p \, dx,$$

for $p \geq 1$. The $L^\infty$ norm is the limit as $p \to \infty$ of the $L^p$ norms, hence the initially peculiar name.