Math 5588 March 9, 2017

So far we have discussed the maximum principle for classical solutions of elliptic equations

$$H(x, u, Du, D^2u) = 0$$

Classical solutions are twice continuously differentiable.

In general, classical solutions do not exist.

Example: The PDE

$$\begin{cases} u_x^2 + u_y^2 = 1, & \text{on } B(0,1) \\ u = 0, & \text{on } \partial B(0,1) \end{cases}$$

does not have a continuously differentiable solution. Here

~~$B(x,r) = \{x \in \mathbb{R}^2 : |x_1^2$~~

$$B(0,1) = \{(x,y) \mid x^2+y^2<1\}.$$

Indeed, if such a solution existed, it could not attain its max or min inside $B(0,1)$, since $u_x = u_y = 0$ at a max or min, but $u_x^2 + u_y^2 = 1$.

Hence max and min both occur on $\partial B(0,1)$ and so $u \equiv 0$, which is a contradiction.

The equation $u_x^2 + u_y^2 = 1$ is a special case of eikonal's equation

$$\begin{cases} |Du| = f, & \text{in } U \\ u = g, & \text{on } \partial U \end{cases}$$

which has many applications

- Control theory / path planning
- Computer vision
- Geometric optics
- Much more...

The eikonal equation is not the only nonlinear PDE for which classical solutions fail to exist; most elliptic (degenerate) nonlinear PDE do not have classical solutions for similar reasons.

The maximum principle suggests a notion of weak solution that can address this problem.

Recall $H(x, z, p, A)$ is *elliptic* if

① $A \leq B \implies H(x,z,p,B) \leq H(x,z,p,A)$

and

② $r \leq s \implies H(x,r,p,A) \leq H(x,s,p,B)$

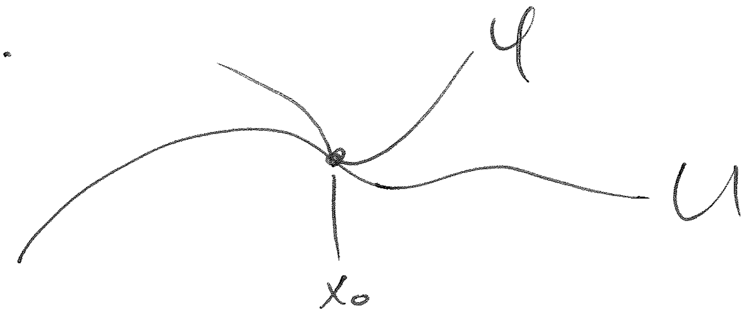
Suppose H is elliptic and u solves

$$H(x, u, Du, D^2u) = 0$$

Let φ be a smooth test function such that for some $x_0 \in U$

$$\varphi(x_0) = u(x_0) \quad \text{and} \quad u(x) \leq \varphi(x)$$

for all $x \in U$.



We say φ touches u from above at x_0. Note: $u-\varphi$ has a *maximum* at x_0, so

$$\nabla u(x_0) = \nabla \varphi(x_0)$$

and
$$\nabla^2 u(x_0) \le \nabla^2 \varphi(x_0)$$

Since H is *elliptic*

$$0 = H(x_0, u(x_0), \nabla u(x_0), \nabla^2 u(x_0)) \ge H(x_0, u(x_0), \nabla \varphi(x_0), \nabla^2 \varphi(x_0))$$

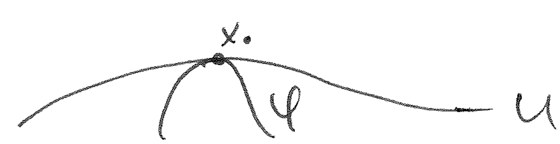
If $u-\varphi$ has a *local* maximum at x_0 then

(1) $$H(x_0, u(x_0), \nabla \varphi(x_0), \nabla^2 \varphi(x_0)) \le 0$$

Similarly, if

$\varphi(x_0) = u(x_0)$ and $u(x) \ge \varphi(x)$ for all x

then φ *touches* u from below at x_0



In this case $u-\varphi$ has a local <u>minimum</u> at x_0 and so

$$Du(x_0) = D\varphi(x_0)$$
$$D^2u(x_0) \geq D^2\varphi(x_0)$$

Since H is <u>elliptic</u>

$$0 = H(x_0, u(x_0), Du(x_0), D^2u(x_0)) \leq H(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0))$$

If $u-\varphi$ has a local minimum at x_0 then

(2) $$H(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0$$

(1) and (2) do not involve <u>derivatives</u> of u and can be used to define a notion of <u>weak</u> solution, called

$\boxed{\text{viscosity solution}}$

<u>Definition</u> (i) A <u>continuous</u> function $u : U \to \mathbb{R}$ is a <u>viscosity subsolution</u> of

$$(*) \qquad H(x, u, Du, D^2u) = 0 \quad \text{in } U$$

if for all $x \in U$ and all smooth test functions $\varphi : U \to \mathbb{R}$ such that

$$u - \varphi \text{ has a local max at } x$$

we have

$$H(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0.$$

(ii) A <u>continuous</u> function $u : U \to \mathbb{R}$ is a <u>viscosity supersolution</u> of $(*)$ if for all $x \in U$ and all φ such that $u - \varphi$ has a local min at x we have

$$H(x, u(x), D\varphi(x), D^2\varphi(x)) \geq 0$$

(iii) A <u>continuous</u> function $u: U \to \mathbb{R}$ is a <u>viscosity solution</u> of (*) if u is both a viscosity sub- and super-solution.

The definition of viscosity solutions should remind you of arguments we made via the maximum principle (i.e., looking at max or min of $u-v$).

It is possible to prove <u>existence</u> and <u>uniqueness</u> of viscosity solutions using maximum principle-type arguments.

<u>Note</u>: The original motivation of viscosity solutions was to regularize first order PDE by adding $\varepsilon \Delta u$:

$$H(x,u,Du) + \varepsilon \Delta u = 0$$

and sending $\varepsilon \to 0$.

$\varepsilon \Delta u$ is called a viscosity term since in fluid equations it represents viscosity of the fluid. The name "viscosity solution" ~~~~ comes from this original motivation by the method of vanishing viscosity, however it is best to think of viscosity solutions in terms of the maximum principle.

Uniqueness of Viscosity Solutions

Thm: Assume $u, v : \overline{U} \to \mathbb{R}$ are continuous and u is a viscosity subsolution of

$$(*) \qquad u + H(x, \nabla u) = 0 \quad \text{in } U$$

and v is a viscosity supersolution of $(*)$. Assume further that u, v, and H are Lipschitz continuous, that is for some

$C > 0$

$$|u(x) - u(y)| \leq C|x-y|$$

$$|v(x) - v(y)| \leq C|x-y|$$

$$|H(x,p) - H(y,p)| \leq C|x-y|.$$

Then if $u \leq v$ on ∂U

then $u \leq v$ in U.

<u>Proof</u>: Since u and v are not differentiable we cannot just look at max of $u-v$ and conclude that

$$Du(x_0) = Dv(x_0).$$

Instead, we "double" the number of variables and consider the max over (x,y) of

$$\Phi_k(x,y) := u(x) - v(y) - k|x-y|^2$$

when k is large, the maximal (x,y) should be close ($|x-y|$ small).

Let
$$M = \max_{\bar{U}} (u-v)$$

and let $(x_k, y_k) \in \bar{U} \times \bar{U}$ such that

$$\Phi_k(x_k, y_k) = \max_{\bar{U}\times\bar{U}} \Phi_k .$$

The proof is split into several steps.

① $\Phi_k(x_k, y_k) \geq M = \max_{\bar{u}} (u-v)$.

Indeed, since $\Phi_k(x,x) = u(x) - v(x) \leq \Phi_k(x_k, y_k)$

① must hold. In particular,

$$\boxed{M \leq \Phi_k(x_k, y_k) \leq u(x_k) - v(y_k)}$$

② Claim $|x_k - y_k| \le \frac{C}{k}$ for some $C > 0$.

To see this, note

$$\Phi_k(x_k, y_k) \ge \Phi_k(x_k, x_k)$$

or

$$u(x_k) - v(y_k) - k|x_k - y_k|^2 \ge u(x_k) - v(x_k)$$

Simplifying we have

$$k|x_k - y_k|^2 \le v(x_k) - v(y_k)$$

$$\le C|x_k - y_k| \quad \text{since } v \text{ Lipschitz.}$$

Thus

$$\boxed{|x_k - y_k| \le \frac{C}{k}}$$

③ Claim $M \le \frac{C}{k}$ for some $C > 0$

and all $k > 0$.

Once we prove this, we send $k \to \infty$ to get

$$\max_{\bar{U}} (u-v) \leq M \leq 0 \quad \text{or} \quad u \leq v.$$

The proof of (3) has 3 steps

(A) If $x_k \in \partial U$ then $u(x_k) \leq v(x_k)$. Thus

$$M \leq u(x_k) - v(y_k) \leq v(x_k) - v(y_k)$$
$$\leq C|x_k - y_k| \quad \text{since } v \text{ Lipschitz}$$
$$\leq \frac{C}{k} \quad \text{by (2)}.$$

(B) If $y_k \in \partial U$ then $u(y_k) \leq v(y_k)$. Thus

$$M \leq u(x_k) - v(y_k) \leq u(x_k) - u(y_k)$$
$$\leq C|x_k - y_k| \leq \frac{C}{k}.$$

(c) Assume $x_k \in \mathcal{U}$ and $y_k \in \mathcal{U}$.

Then

$$x \longmapsto u(x) - v(y_k) - k|x - y_k|^2$$

has a maximum at $x = x_k$, that is

$u - \varphi$ has a max at x_k when

$$\varphi(x) = v(y_k) + k|x - y_k|^2$$

Note: $\nabla \varphi(x_k) = 2k(x_k - y_k) =: p_k$.

Therefore

$$u(x_k) + H(x_k, p_k) \leq 0 \qquad (1)$$

Similarily $\quad y \longmapsto u(x_k) - v(y) - k|x_k - y|^2$

has a maximum at $y = y_k$. That is

$\psi - v$ has a max at y_k

where $\quad \psi(y) = u(x_k) - k|x_k - y|^2$

In other words

$v - \varphi$ has a minimum at y_k

and $\nabla\varphi(y_k) = 2k(x_k - y_k) = p_k$.

Hence

$$v(y_k) + H(y_k, p_k) \geq 0 \qquad (2)$$

Subtracting (1) from (2)

$$\begin{aligned} M \leq u(x_k) - v(y_k) &\leq H(y_k, p_k) - H(x_k, p_k) \\ &\leq C|x_k - y_k| \\ &\leq \frac{C}{k} \end{aligned}$$

This completes the proof. ▨