Math 5588 | 3/21/17

The Wave Equation

The wave equation in $\mathbb{R}^n$ is

$$(W) \quad \begin{cases} u_{tt} - \Delta u = 0 & , x \in \mathbb{R}^n, t > 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

The PDE is <u>not</u> elliptic so the maximum principle is not useful here. The wave equation (W) is <u>hyperbolic</u>.

<u>n=1</u>: Vibrating String

<u>n=2</u>: Vibrating membrane

<u>n=3</u>: Sound waves, electromagnetic radiation.

$n \geq 4$: • Mathematical curiosity
• String theory in physics ...

when $n=1$ we have d'Alembert's formula

$$\boxed{u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds}$$

The aim now is to derive a similar formula for $n \geq 2$.

We use the method of spherical means

Fix $x \in \mathbb{R}^n$, $t>0$, $r>0$ and define

$$U(x;r,t) := \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y,t)\,dS(y)$$

$U(x;r,t)$ is the average of $u(x,t)$

over the boundary $\partial B(x,r)$ of the

ball $B(x,r)$. Here

$$|\partial B(x,r)| = \text{surface area of boundary of ball}$$

$$= n\alpha(n) r^{n-1}$$

where $\alpha(n)$ = volume of $B(0,1)$. in $\mathbb{R}^n$

It turns out if we fix $x \in \mathbb{R}^n$

then

$$(r,t) \longmapsto U(x;r,t)$$

solves a 1-D wave equation for which

we can apply d'Alembert's formula.

Then we recover $u(x,t)$ by sending $r \to 0$:

$$u(x,t) = \lim_{r \to 0^+} U(x;r,t)$$

We define

$$F(x;r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} f(y)\, dS(y)$$

and

$$G(x;r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} g(y)\, dS(y).$$

<u>Lemma</u> (Euler-Poisson-Darboux equation)

$U(x;r,t)$ solves
$$\begin{cases} U_{tt} - U_{rr} - \left(\frac{n-1}{r}\right) U_r = 0, & t>0,\ r>0 \\ U(x;r,0) = F, \quad U_t(x;r,0) = G \end{cases}$$

<u>Proof</u>: Making a change of variables

$$z = \frac{y-x}{r}, \quad dS(z) = \frac{dS(y)}{r^{n-1}}$$

we have

$$U(x;r,t) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(0,1)} u(x+rz)\, r^{n-1}\, dS(z)$$

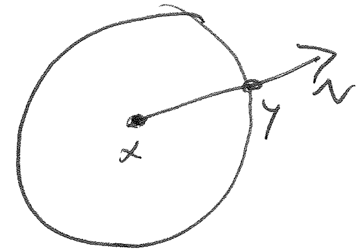
$$= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz)\, dS(z)$$

Hence

$$(*) \quad U_r(x;r,t) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \nabla u(x+rz)\cdot z\, dS(z)$$

$$= \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \nabla u(y)\cdot\left(\frac{y-x}{r}\right) dS(y)$$

$$= \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y)\, dS(y) \qquad \left(\nu = \frac{y-x}{r} = \text{outward normal}\right)$$



$$= \frac{1}{|\partial B(x,r)|} \int_{B(x,r)} \Delta u(y,t)\, dy$$

where we used integration by parts, or Green's identity in last step. So

$$\boxed{U_r(x;r,t) = \frac{1}{|\partial B(x,r)|} \int_{B(x,r)} \Delta u(y,t)\, dy}$$

Note $\quad |\partial B(x,r)| = n\alpha(n) r^{n-1} = \frac{n}{r}\alpha(n) r^n$

$$= \frac{n}{r} |B(x,r)|$$

So $$U_r = \frac{r}{n} \cdot \frac{1}{|B(x,r)|} \int_{B(x,r)} \Delta u(y,t)\, dy$$

Hence
$$\boxed{\lim_{r \to 0^+} U_r(x;r,t) = 0}$$

Furthermore

$$U_r(x;r,t) = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} u_{tt}(y,t)\, dy$$

and so

$$\frac{\partial}{\partial r}\left(r^{n-1} U_r(x;r,t)\right) = \frac{1}{n\alpha(n)} \frac{\partial}{\partial r} \int_{B(x,r)} u_{tt}(y,t)\, dy$$

$$\cancel{= \frac{1}{n\alpha(n)} \frac{\partial}{\partial r} \int_{B(0,1)} u_{tt}(x+rz,t)\, r^n\, dz}$$

$$= \frac{1}{n\alpha(n)} \frac{\partial}{\partial r} \overbrace{\int_0^r \int_{\partial B(x,s)} u_{tt}(y,t)\, dS(y)\, ds}^{\text{Polar coord.}}$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}(y,t)\, dS(y)$$

$$= r^{n-1} \cdot \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u_{tt}(y,t)\, dS(y)$$

$$= r^{n-1}\, U_{tt}(x;r,t)$$

So

$$\frac{\partial}{\partial r}\left(r^{n-1} U_r \right) = r^{n-1} U_{tt}$$

$$(n-1) r^{n-2} U_r + r^{n-1} U_{rr} = r^{n-1} U_{tt}$$

Divide by r^{n-1} to get

$$U_{tt} = U_{rr} + \left(\frac{n-1}{r}\right) U_r \qquad \boxed{/\!/\!/}$$

When $\underline{n=3}$ we can use a fancy trick to convert this into the usual wave equation. Define

$$\tilde{u}(r,t) = r\,U(r,t)$$

$$\tilde{F}(r,t) = r\,F(r,t)$$

$$\tilde{G}(r,t) = r\,G(r,t)$$

We claim $\tilde{u}(r,t)$ solves

$$(*) \quad \begin{cases} \tilde{u}_{tt} - \tilde{u}_{rr} = 0 \quad , \quad r>0,\ t>0 \\ \tilde{u}(r,0) = \tilde{F} \ , \ \tilde{u}_t(r,0) = \tilde{G} \\ \tilde{u}(0,t) = 0 \end{cases}$$

To see this write

$$\tilde{u}_{tt} = r u_{tt}$$

$$= r\left(u_{rr} + \frac{n-1}{r} u_r\right)$$

$$= r u_{rr} + (n-1) u_r$$

$$= \frac{\partial}{\partial r}(r u_r + u) \qquad \text{if } n=3$$

so $n-1=2$.

And $\tilde{u}_{rr} = \frac{\partial}{\partial r} \tilde{u}_r = \frac{\partial}{\partial r}(r u_r + u)$

Hence $\tilde{u}_{tt} = \tilde{u}_{rr}$ when $\underline{n=3}$.

Now, we will use d'Alembert to solve (*).

We use the method of <u>odd</u> reflection.

For $r<0$ define

$$\tilde{G}(r,\cancel{t}) = -\hat{G}(-r)$$

$$\tilde{F}(r) = -\hat{F}(-r)$$

We then solve the wave equation

on the entire real line

$$\begin{cases} v_{tt} - v_{rr} = 0 , & t>0,\ -\infty<r<\infty \\ v(r,0) = \tilde{F}(r), \quad v_t(r,0) = \tilde{G}(r) \end{cases}$$

Since $\tilde{F}(0) = 0 = \tilde{G}(0)$ and $\widehat{F}$ and $\widehat{G}$

are odd functions, the solution $v(r,t)$

will be an odd function of r

(by d'Alembert's formula) and so

$$V(0,t) = 0 \qquad \text{for all} \quad t$$

Hence $\quad V(r,t) = \tilde{\hat{u}}(r,t)$.

By d'Alembert's formula

$$\tilde{\hat{u}}(r,t) = \frac{1}{2}\left(\hat{F}(r+t) + \tilde{F}(r-t)\right) + \frac{1}{2}\int_{r-t}^{r+t} \tilde{\hat{G}}(s)\,ds.$$

Let's take $\boxed{0 \le r < t}$. Then $\quad r-t<0$

so $\qquad \tilde{\hat{F}}(r-t) = -\hat{F}(t-r)$

and $\displaystyle \int_{r-t}^{0} \tilde{\hat{G}}(s)\,ds = \int_{r-t}^{0} -\hat{G}(-s)\,ds$

$$\tau = -s \qquad = \int_{t-r}^{0} \tilde{G}(\tau)\,d\tau$$

Hence

$$\int_{r-t}^{r+t} \tilde{G}(s)\,ds = \int_{t-r}^{t+r} \tilde{G}(s)\,ds \quad \text{and so}$$

$$\tilde{U}(r,t) = \frac{1}{2}\left(\tilde{F}(r+t) - \hat{F}(t-r)\right) + \frac{1}{2}\int_{t-r}^{t+r} \hat{G}(s)\,ds \quad \text{for } 0 \le r < t$$

Now,

$$u(x,t) = \lim_{r \to 0^+} U(x;r,t)$$

$$= \lim_{r \to 0^+} \frac{\tilde{U}(r,t)}{r}$$

$$= \lim_{r \to 0} \frac{\tilde{F}(t+r) - \tilde{F}(t-r)}{2r} + \lim_{r \to 0} \frac{1}{2r}\int_{t-r}^{t+r} \tilde{G}(s)\,ds$$

$$= \tilde{F}'(t) + \tilde{G}(t)$$

$$= \frac{\partial}{\partial t}\Big(t F(t)\Big) + t G(t)$$

$$= F(t) + tG(t) + t F'(t)$$

$$F(t) = \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} f(y)\, ds(y)$$

$$z = \frac{y-x}{t}$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} f(x+tz)\, ds(z)$$

and

$$F'(t) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \nabla f(x+tz)\cdot z\, dS(z)$$

$$= \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} \nabla f(y) \cdot \left(\frac{y-x}{t}\right) dS(y)$$

So

$$t F'(t) = \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} \nabla f(y) \cdot (y-x) \, dS(y)$$

and finally we have

$$\boxed{u(x,t) = \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} t g(y) + f(y) + \nabla f(y) \cdot (y-x) \, dS(y)}$$

This is called Kirchoff's formula for the solution $u(x,t)$ of wave eq.

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^3, \; t > 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x). \end{cases}$$

What about $c \neq 1$ wave speeds $(c>0)$?

let $u(x,t)$ solve

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, & x \in \mathbb{R}^3, \ t>0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

let $v(x,t) = u(x, \frac{t}{c})$. Then

$$V_{tt}(x,t) = \frac{1}{c^2} u_{tt}(x, \tfrac{t}{c})$$

$$\Delta v(x,t) = \Delta u(x, \tfrac{t}{c}) = \frac{1}{c^2} u_{tt} = v_{tt}$$

and $v(x,0) = f(x)$, $v_t(x,0) = \frac{1}{c} u_t(x,0) = \frac{g(x)}{c}$

By Kirchoff's formula

$$\boxed{\begin{aligned} u(x,t) &= v(x, ct) \\ &= \frac{1}{|\partial B(x,ct)|} \int_{\partial B(x,ct)} t g(y) + f(y) + Df(y)\cdot(y-x) \, dS(y) \end{aligned}}$$

This says the solution $u(x,t)$ at (x,t) depends only on the initial data f and g on $\partial B(x, ct)$,

that is, $u(x,t)$ depends on $f(y)$ and $g(y)$ only for $y \in \mathbb{R}^n$ such that

$$|y-x| = ct$$

These are exactly all the $y \in \mathbb{R}^n$ s.t. a sound wave starting at y at $t=0$ and travelling at speed c will reach ~~4~~ x at exactly time t.

This is Huygen's principle

When $n=3$, information (or disturbances) in the wave equation travel at exactly speed c.

This is what makes verbal communication and music possible in our 3 dimensional world.

We will see next lecture that this is not the case in even dimensions (or $n=1$). As a remark, it is possible, but much harder, to use the same ideas to derive a representation formula for odd $n \geq 5$. We get

$$u(x,t) = \frac{1}{\gamma_n}\left[\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(\frac{t^{n-2}}{|\partial B(x,t)|}\int_{\partial B(x,t)} f(y)\,dS(y)\right)\right.$$

$$\left. + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(\frac{t^{n-2}}{|\partial B(x,t)|}\int_{\partial B(x,t)} g(y)\,dS(y)\right)\right]$$

where $\gamma_n = 1 \cdot 3 \cdot 5 \cdots (n-2)$

and $c = 1$. As an exercise, make a change of variables to extend this formula to $c \neq 1$.