Math 5588 | 3/23/17

The Wave Equation

Last time we derived Kirchoff's formula

$$u(x,t) = \frac{1}{|\partial B(x,ct)|}\int_{\partial B(x,ct)} t\,g(y) + f(y) + \nabla f(y)\cdot(y-x)\; dS(y)$$

for the solution of the wave equation

$$(W)\begin{cases} u_{tt} - c^2\Delta u = 0, & x\in\mathbb{R}^n,\ t>0\\ u(x,0) = f(x) \\ u_t(x,0) = g(x)\end{cases}$$

in $n=3$ dimensions.

For $\underline{n=2}$ we use the "method of descent"

from $n=3$ dimension.

Assume $\boxed{c=1}$ and

$$u(x_1,x_2,t) = u(x,t)$$

solves (w) in $n=2$ dimensions. Define

$$\bar{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t)$$

$$\bar{f}(x_1,x_2,x_3) = f(x_1,x_2)$$

$$\bar{g}(x_1,x_2,x_3) = g(x_1,x_2)$$

Then $\bar{u}$ satisfies

$$(\bar{w}) \begin{cases} \bar{u}_{tt} - \Delta \bar{u} = 0, & \text{in } \mathbb{R}^3 \times (0,\infty) \\ \bar{u}(x,0) = \bar{f}(x) \\ \bar{u}_t(x,0) = \bar{g}(x). \end{cases}$$

Write $x = (x_1,x_2) \in \mathbb{R}^2$ and

$\bar{x} = (x_1,x_2,0) \in \mathbb{R}^3$

By an earlier form of Kirchhoff's formula

$$u(x,t) = \bar{u}(\bar{x},t)$$

$$= \frac{\partial}{\partial t}\left(\frac{t}{|\partial \bar{B}(\bar{x},t)|}\int_{\partial \bar{B}(\bar{x},t)} \bar{f}\, d\bar{S}\right) + \frac{t}{|\partial \bar{B}(\bar{x},t)|}\int_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S}$$

where $\bar{B}(\bar{x},t)$ denotes the ball in $\mathbb{R}^3$ with center $\bar{x}$ and radius $t > 0$.

$d\bar{S}$ = two-dimensional surface measure on $\partial \bar{B}(\bar{x},t)$

We can parametrize $\int_{\partial \bar{B}(\bar{x},t)} \bar{f}\, d\bar{S}$ and $\int_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S}$ via $(y, \gamma(y)) \in \mathbb{R}^3$ where

$$y \in B(x,t) \subseteq \mathbb{R}^2 \quad \text{and} \quad \gamma(y) = \sqrt{t^2 - |x-y|^2}$$

Then

$$|(y, \gamma(y)) - (x, 0)|^2$$

$$= |y - x|^2 + \gamma(y)^2 = t^2$$

The surface area element $d\bar{S}$ is

$$d\bar{S}(y) = \sqrt{1 + |\nabla \gamma(y)|^2}\, dy$$

Since $\bar{f}$ is an even function of x_3

$$\frac{1}{|\partial \bar{B}(\bar{x}, t)|} \int_{\partial \bar{B}(\bar{x}, t)} \bar{f}\, d\bar{S} = \frac{1}{4\pi t^2} \int_{\partial \bar{B}(\bar{x}, t)} \bar{f}\, d\bar{S}$$

$$= \frac{2}{4\pi t^2} \int_{B(x,t)} f(y) \sqrt{1 + |\nabla \gamma(y)|^2}\, dy$$

Note

$$\frac{\partial \gamma}{\partial y_i} = \frac{1}{2}\left(t^2 - |x - y|^2\right)^{-\frac{1}{2}} \cdot \left(-2(y_i - x_i)\right)$$

$$= \frac{x_i - y_i}{\sqrt{t^2 - |x - y|^2}}$$

Thus

$$|\nabla \gamma(y)|^2 = \frac{(x_1-y_1)^2 + (x_2-y_2)^2}{\left(\sqrt{t^2-|x-y|^2}\right)^2}$$

$$= \frac{|x-y|^2}{t^2-|x-y|^2}$$

Hence

$$1+|\nabla \gamma(y)|^2 = \frac{t^2-|x-y|^2+|x-y|^2}{t^2-|x-y|^2}$$

$$= \frac{t^2}{t^2-|x-y|^2}$$

and we have

$f(y)$

$$\frac{1}{|\partial \bar{B}(\bar{x},t)|}\int_{\partial \bar{B}(\bar{x},t)} \bar{f}\, d\bar{s} = \frac{1}{2\pi t^2}\int_{B(x,t)} \frac{t\, g(y)}{\sqrt{t^2-|x-y|^2}}\, dy$$

$f(y)$

$$= \frac{1}{2\pi t}\int_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|x-y|^2}}\, dy$$

The same argument applies to other integral and we have

$$u(x,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi} \int_{B(x,t)} \frac{\cancel{g} f(y)}{\sqrt{t^2 - |x-y|^2}}\, dy \right)$$

$$+ \frac{1}{2\pi} \int_{B(x,t)} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}}\, dy$$

Make a change of variables in first integral

$$z = \frac{y-x}{t}, \quad dy = t^2\, dz$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2\pi} \int_{B(x,t)} \frac{f(y)}{\sqrt{t^2 - |x-y|^2}}\, dy \right) = \frac{\partial}{\partial t}\left(\frac{t}{2\pi} \int_{B(0,1)} \frac{f(x+zt)}{\sqrt{1-|z|^2}}\, dz \right)$$

$$= \frac{1}{2\pi} \int_{B(0,1)} \frac{f(x+zt)}{\sqrt{1-|z|^2}}\, dz + \frac{t}{2\pi} \int_{B(0,1)} \frac{\nabla f(x+zt)\cdot z}{\sqrt{1-|z|^2}}\, dz$$

$$= \frac{1}{2\pi} \int_{B(0,1)} \frac{f(x+zt) + t\, Df(x+zt)\cdot z}{\sqrt{1-|z|^2}}\, dz$$

$$z = \frac{y-x}{t}$$

$$= \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{f(y) + t\, Df(y)\cdot\left(\frac{y-x}{t}\right)}{\sqrt{1 - \frac{|x-y|^2}{t^2}}}\, dy$$

$$= \frac{1}{2\pi t} \int_{B(x,t)} \frac{f(y) + Df(y)\cdot(y-x)}{\sqrt{t^2 - |y-x|^2}}\, dy$$

$$= \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t f(y) + t\, Df(y)\cdot(y-x)}{\sqrt{t^2 - |x-y|^2}}\, dy.$$

Therefore

$$\boxed{u(x,t) = \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t f(y) + t\, Df(y)\cdot(y-x) + t^2 g(y)}{\sqrt{t^2 - |x-y|^2}}\, dy}$$

This is Kirchhoff's formula in $n=2$ dimensions.

If $c \neq 1$, suppose $u(x,t)$ solves

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, \ x \in \mathbb{R}^2, \ t > 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Set $v(x,t) = u(x, \frac{t}{c})$

Then as before
$$v_{tt} - \Delta v = 0$$
$$v(x,0) = f(x)$$
$$v_t(x,0) = \frac{g(x)}{c}$$

Thus

$$\boxed{\begin{aligned} u(x,t) &= v(x, ct) \\ &= \frac{1}{2\pi c^2 t^2} \int_{B(x,ct)} \frac{ct f(y) + ct \nabla f(y) \cdot (y-x) + c^2 t^2 g(y)}{\sqrt{c^2 t^2 - |x-y|^2}} \, dy \end{aligned}}$$

Notice $u(x,t)$ depends on f and g

inside $B(x, ct)$. So $u(x,t)$ is an

~~the~~ average of all sounds created

at points $y \in \mathbb{R}^2$ and time $t=0$

for

$$|y-x| \leq ct$$

This means disturbances at $(y,0)$ affect the solution at all future times

$$t \geq \frac{|y-x|}{c}$$

So disturbances travel at speed c

and all speeds slower than c.

In other words you would hear a sound forever, though its effect diminishes over time. Another way to think of it is you hear an average of all sounds created at time $t=0$ in the ball $B(x, ct)$.

Thus, if we lived in a flatworld of $n=2$ dimensions, music and verbal communication would be <u>much</u> different.

This is Huygen's Principle in $n=2$ dimensions. <u>For even</u> $n \geq 4$,

$$u(x,t) = \frac{1}{\gamma_n}\left[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(\frac{t^n}{|B(x,t)|}\int_{B(x,t)} \frac{f(y)}{\sqrt{t^2-|x-y|^2}}\,dy\right)\right.$$

$$\left.+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(\frac{t^n}{|B(x,t)|}\int_{B(x,t)} \frac{g(y)}{\sqrt{t^2-|x-y|^2}}\,dy\right)\right]$$

where $\gamma_n = 2\cdot 4 \cdots (n-2)\cdot n$.

Energy methods for wave equation

Consider the boundary-value problem

$$(W)\quad \begin{cases} u_{tt} - \Delta u = 0, & \text{in } U \times (0,T] = U_T \\ u = g, & \text{on } \Gamma_T = \overline{U_T} \setminus U_T \\ u_t = h, & \text{on } U \times \{t=0\}. \end{cases}$$

when $U \subset \mathbb{R}^n$ is open and bounded.

Thm (Uniqueness)
There exists at most one function satisfying (W)

Proof: If v is another solution then $w = u - v$ satisfies

$$\begin{cases} w_{tt} - \Delta w = 0, & \text{in } U_T \\ w = 0, & \text{on } \Gamma_T \\ w_t = 0, & \text{on } U \times \{t=0\}. \end{cases}$$

Define the energy

$$E(t) := \frac{1}{2}\int_U w_t(x,t)^2 + |\nabla w(x,t)|^2 \, dx$$

Then

$$\frac{dE}{dt} = \int_U w_t w_{tt} + \nabla w \cdot \nabla w_t \, dx$$

Integration by parts.

$$= \int_U w_t w_{tt} - (\Delta w) w_t \, dx$$

$$= \int_U w_t (w_{tt} - \Delta w) \, dx = 0$$

Hence $E(t) = 0 = E(0)$ and $w = u - v = 0$. $\blacksquare$

Domain of dependence.

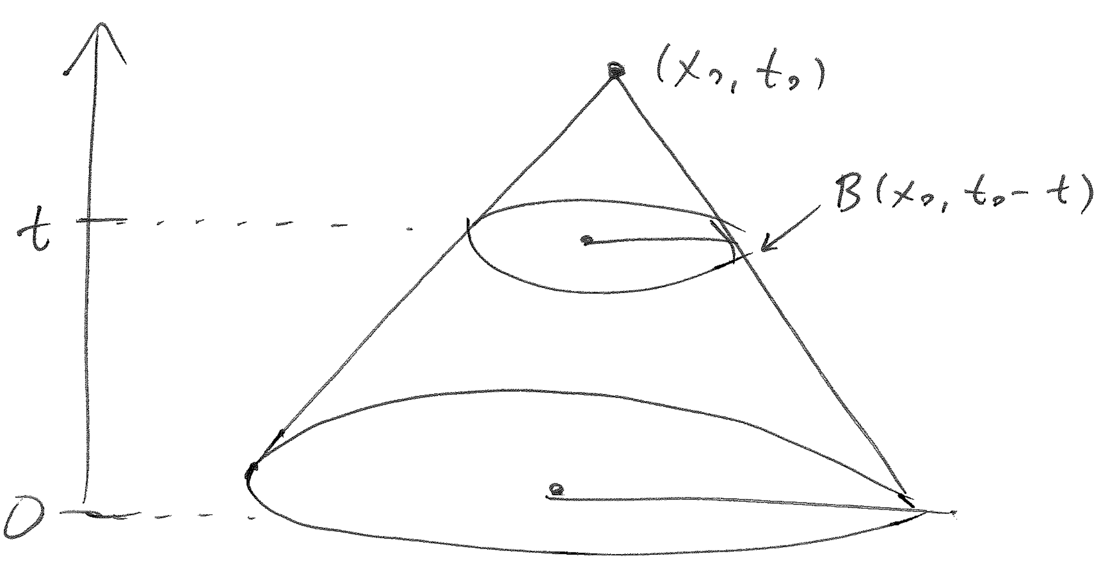
Without solving wave equation we can deduce the domain of dependence via simple energy method techniques.

Suppose $u(x,t)$ solves

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

and for $x_0 \in \mathbb{R}^n$, $t_0 > 0$ consider the backward wave cone

$$K(x_0,t_0) = \{(x,t) \mid 0 \le t \le t_0,\ |x - x_0| \le t_0 - t\}$$



Theorem: If $u \equiv u_t \equiv 0$ on

$$B(x_0, t_0) \times \{t = 0\}$$

then $u \equiv 0$ in the cone $K(x_0, t_0)$

Proof: Define the energy

$$e(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} u_t(x,t)^2 + |\nabla u(x,t)|^2 \, dx$$

$$= \underbrace{\frac{1}{2} \int_0^{t_0 - t} \int_{\partial B(x_0, \tau)} u_t^2 + |\nabla u|^2 \, dS(x) \, d\tau}_{\text{polar coordinates.}}$$

Then

$$\frac{de}{dt} = -\frac{1}{2} \int_{\partial B(x_0, t_0 - t)} u_t^2 + |\nabla u|^2 ds + \int_{B(x_0, t_0 - t)} u_t u_{tt} + \nabla u \cdot \nabla u_t \, dx$$

by chain rule.

Integrating by parts in second term we have

$$\frac{de}{dt} = -\frac{1}{2}\int_{\partial B(x_0, t_0-t)} u_t^2 + |Du|^2 dS + \underbrace{\int_{B(x_0,t_0-t)} u_t u_{tt} - u_t \Delta u \, dx}_{=0}$$

$$+ \int_{\partial B(x_0, t_0-t)} u_t \frac{\partial u}{\partial \nu} dS$$

$$= \int_{\partial B(x_0, t_0-t)} \frac{\partial u}{\partial \nu} u_t - \frac{1}{2} u_t^2 - \frac{1}{2}|Du|^2 dS$$

Since $u_{tt} - \Delta u = 0$.

<u>Note</u>

$$\left|\frac{\partial u}{\partial \nu} u_t\right| \leq |u_t|\,|Du| \leq \frac{1}{2} u_t^2 + \frac{1}{2}|Du|^2$$

Cauchy Schwarz

Cauchy's inequality

$$\left|\frac{\partial u}{\partial \nu}\right| = |Du \cdot \nu| \leq |Du|\,|\nu| = |Du|$$

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$$

Therefore $\frac{de}{dt} \leq 0$.

Since $e(0) = 0$ and $e(t) \geq 0$

for all t we have $e(t) = 0$

and hence $u = 0$ in $K(x_0, t_0 - t)$. $\blacksquare$