Math 5588 4/13/2017

Hamilton-Jacobi Equations

Recall: Scalar conservation law

$$(*)\begin{cases} u_t + \frac{\partial}{\partial x} F(u) = 0, & -\infty < x < \infty, \ t > 0 \\ u(x,0) = f(x) \end{cases}$$

Weak solutions satisfy the entropy condition if

$$F'(u_L) > F'(u_R)$$

across shock curves. If F is strictly convex ($F'' > 0$) this is equivalent to

$$(E) \quad \boxed{u_L > u_R \quad \text{along shocks}}$$

However, (E) is not so useful mathematically as it requires $u(x,t)$ to have a specific form (piecewise smooth, discontinuity along shock curve).

A more general way to formulate the entropy condition is to ask that for some $C > 0$

$$(E') \quad \boxed{u(x+z,t) - u(x,t) \leq C\left(1+\frac{1}{t}\right)z}$$

for all $x, z \in \mathbb{R}$, $t > 0$, $z > 0$

Note	$z > 0$

(E') makes no reference to a shock curve, but enforces the same (or similar) entropy condition. Indeed, if (E') holds then along a shock we have

$$u_l \approx u(x,t)$$

$$u_r \approx u(x+z,t), \quad z > 0$$

Sending $t \to 0^+$ we have

$$u_R - u_\ell \leq 0$$

which is exactly (E). The $\frac{1}{t}$ factor ensures that (E') says nothing at $t=0$, since we could have initial conditions with shocks in either direction.

<u>Fact</u>: If F is <u>convex</u> and u is a weak solution satisfying (E') then u is <u>unique</u>.

For proof (which is hard) see Evans PDE book chapter 3.4.

Since
$$u_t + F'(u) u_x = 0$$
is not necessarily <u>increasing</u> in u, the maximum principle (and thus viscosity solutions) are not applicable. However, there is a clever trick to get around this. Define

$$V(x,t) = \int_0^x u(s,t)\, ds$$

Then $V_x(x,t) = u(x,t)$ and so

$$\frac{\partial}{\partial t}(V_x) + \frac{\partial}{\partial x} F(V_x) = 0$$

Switching the order of derivatives in first term we have

$$\frac{\partial}{\partial x}\left(V_t + F(V_x)\right) = 0$$

Therefore

$$V_t + F(V_x) = \text{Constant}$$

We also have

$$V(x,0) = \int_0^x f(s)\,ds =: g(x)$$

Picking <u>Constant = 0</u>

$$(H) \quad \begin{cases} V_t + F(V_x) = 0 \\ \qquad V(x,0) = g(x) \end{cases}$$

(H) is called a Hamilton–Jacobi equation

Notice V does not show up in (H)

So (H) is automatically <u>degenerate</u> <u>elliptic</u>

so the maximum principle and viscosity solution theory apply to (H).

Conversely, if v solves (H) then

$$\frac{\partial}{\partial x}\left(v_t + F(v_x)\right) = 0$$

or

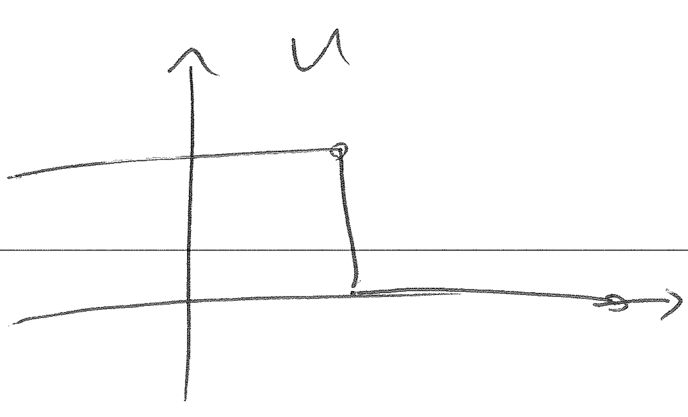
$$\frac{\partial}{\partial t}(v_x) + F'(v_x)\, v_{xx} = 0$$

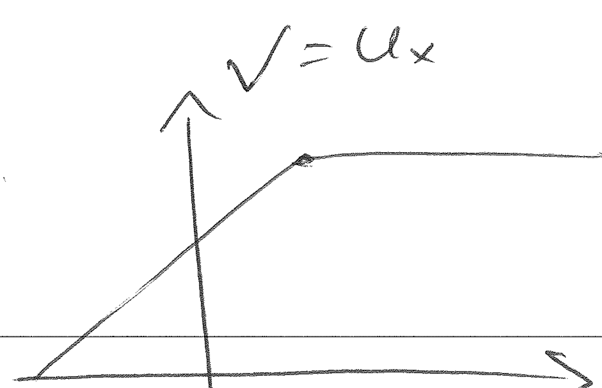
So $u = v_x$ solves the scalar conservation law

$$u_t + F'(u)\, u_x = 0$$

$$\left(u_t + \frac{\partial}{\partial x} F(u)\right) = 0$$

Since u may have discontinuities, v is <u>continuous</u> but $v_x = u$ has discontinuities





The entropy condition

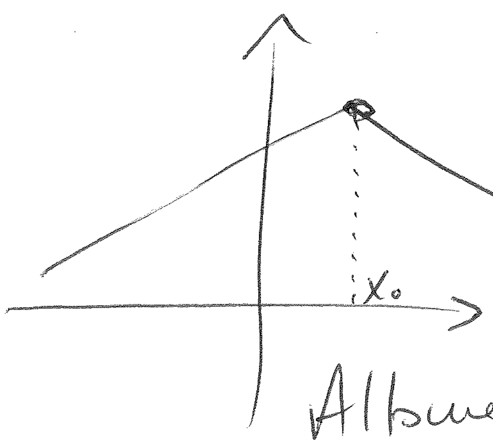
$$u(x+z,t) - u(x,t) \le C\left(1+\frac{1}{t}\right) z$$

becomes

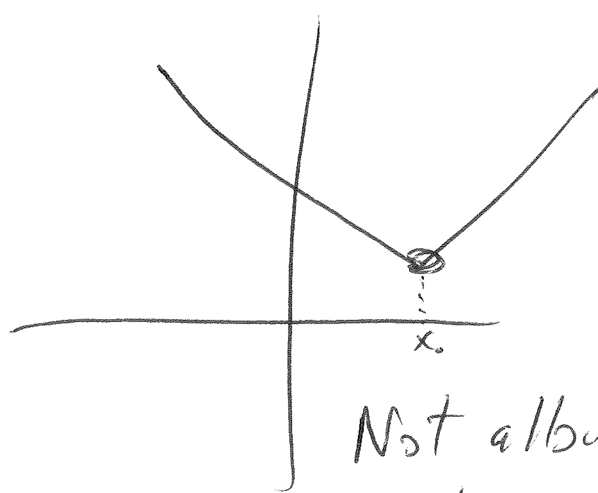
$$(5) \quad V_x(x+z,t) - V_x(x,t) \le C\left(1+\frac{1}{t}\right) z$$

So if V_x has a discontinuity at x_0 then

$$\lim_{x\to x_0^+} V_x(x,t) \le \lim_{x\to x_0^-} V_x(x,t)$$



Allowed by Entropy Cond.



Not allowed by Entropy Cond.

Fact: If $u(x,t)$ is the entropy solution of (*) then $V(x,t) = \int_0^x u(s,t)\,ds$ is the viscosity solution of (H).

Hamilton - Jacobi Equations

Let us write a general Hamilton-Jacobi equation as

$$(H) \begin{cases} u_t + H(u_x) = 0, & -\infty < x < \infty, \ t > 0 \\ u(x,0) = f(x). \end{cases}$$

Applications:

(1) Optimal control
(2) Classical mechanics (physics)
(3) Computational fluid dynamics
→ level set method

... many more.

Let's first look at simple solutions of (H), namely affine solutions

$$u(x,t) = a + px + ct$$

Then $u_x = p$, $u_t = c$ and so

$$c + H(p) = 0 \quad \text{or} \quad c = -H(p).$$

Hence $u(x,t) = a + px - H(p)t$

solves (H) with $u(x,0) = a + px$.

Suppose we want $\underline{u(y,0) = f(y)}$

for some y. Then

$$f(y) = u(y,0) = a + py$$

and so $a = f(y) - py$. Hence

$$\boxed{u(x,t) = f(y) + (x-y)p - H(p)t}$$

solves (H) and satisfies $u(y,0)=f(y)$

<u>Note</u> y and p are free parameters.

We can form a so-called <u>envelope</u> solution from $u(x,t)$:

$$(*) \quad \boxed{u(x,t) = \min_{y\in\mathbb{R}} \max_{p\in\mathbb{R}} \left\{ f(y) + (x-y)p - H(p)t \right\}}$$

It turns out $(*)$ is the solution (in viscosity sense) of (H). This is <u>not</u> obvious yet.

Note that

$$u(x,t) = \min_{y\in\mathbb{R}} \left\{ f(y) + \max_{p\in\mathbb{R}} \left\{ (x-y)p - H(p)t \right\} \right\}$$

$$= \min_{y\in\mathbb{R}} \left\{ f(y) + t \max_{p\in\mathbb{R}} \left\{ \left(\frac{x-y}{t}\right)p - H(p) \right\} \right\}$$

Def] We define the Legendre Transform of H to be

$$H^*(q) = \max_{p \in \mathbb{R}} \{ qp - H(p) \}$$

Then (*) becomes

$$\boxed{u(x,t) = \min_{y \in \mathbb{R}} \left\{ f(y) + tL\left(\frac{x-y}{t}\right) \right\}} \quad (**)$$

where $L = H^*$. The formula (**) is called the Hopf-Lax formula.

We aim to prove that the Hopf-Lax formula solves (H), but this will take some work.

The Legendre Transform

Let us get more familiar with the Legendre transform. Recall

$$H^*(q) = \max_{p \in \mathbb{R}} \{ pq - H(p) \}$$

[Note: We always assume max exists, otherwise replace max by sup.].

Note that $H^*(q) = pq - H(p)$

where $q - H'(p) = 0$

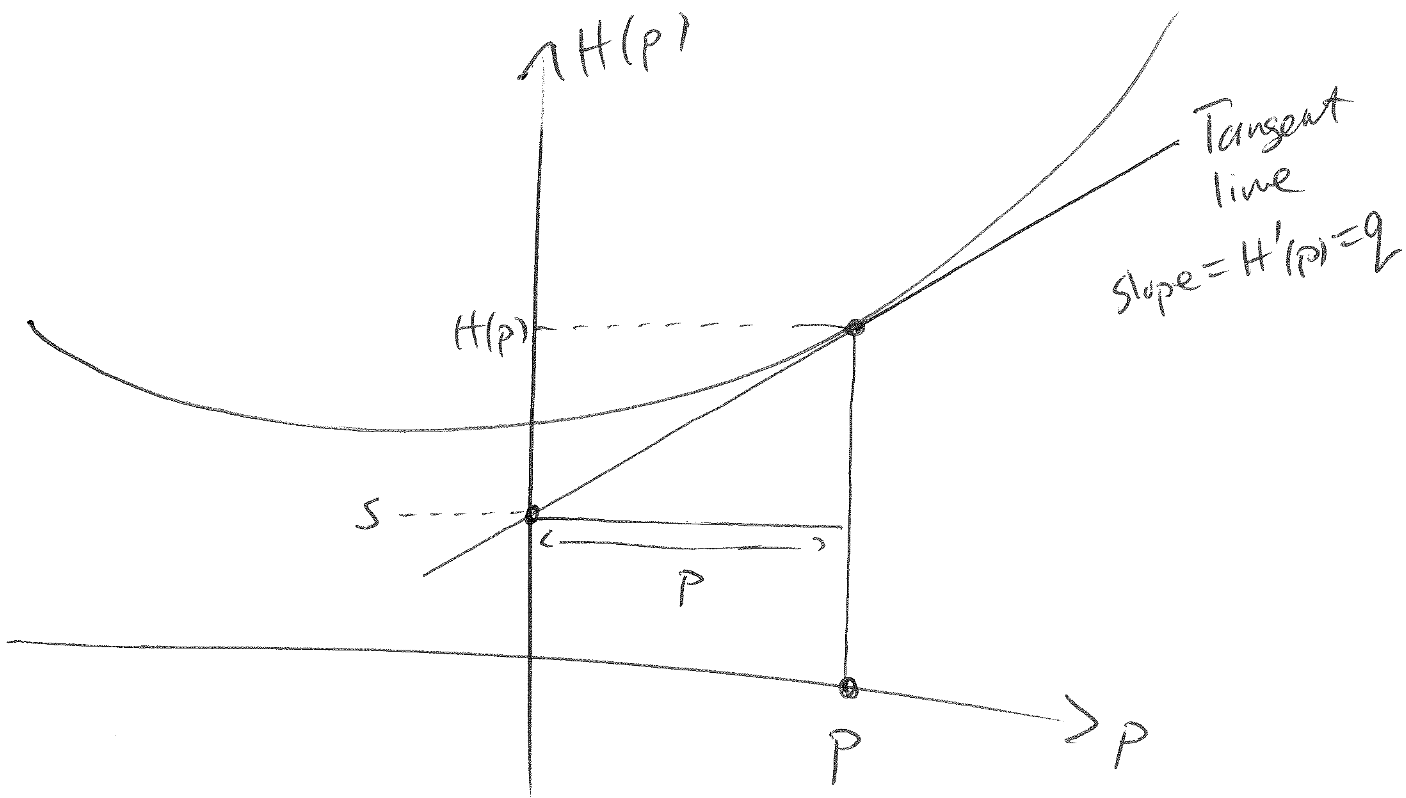
or $q = H'(p)$

Therefore
$$\begin{cases} H^*(q) = pH'(p) - H(p) \\ \text{where } q = H'(p) \end{cases}$$

<u>Note</u> $q = H'(p)$ is uniquely solvable for $p = p(q)$ when H' is <u>strictly</u> <u>increasing</u>

or $H'' > 0$. We from now on assume H is <u>strictly</u> <u>convex</u> or

$$\boxed{H'' > 0}$$



$$Y \text{ intercept of Tangent} = s = H(p) - pH'(p) = -H^*(q)$$

Thus

$$H^*(q) = -\left(\begin{array}{c}\text{Y-intercept of tangent line to } H \\ \text{with slope } q\end{array}\right)$$

Therefore

$T(p) = pq - H^*(q)$ is the tangent line to $H(p)$ with <u>slope</u> q.

Since H is <u>convex</u>

$$H(p) = \max_{q \in \mathbb{R}} \{ pq - H^*(q) \}$$

In other words $\boxed{H = H^{**}.}$

If H convex, then taking Legendre transform twice gives H back.

Our argument was not quite rigorous since the <u>max</u> in the Legendre transform may not exist.

$$H^*(q) = \max_{p \in \mathbb{R}} \{ pq - H(p) \}$$

$$H(p) = \max_{q \in \mathbb{R}} \{ pq - H^*(q) \}.$$

To ensure the <u>max exists</u> we need to also assume

$$(*) \quad \boxed{\lim_{|p| \to \infty} \frac{H(p)}{|p|} = \infty}$$

That is, H has <u>superlinear</u> <u>growth</u>

If $(*)$ holds then $pq - H(p)$ cannot be made arbitrarily large by choosing $|p|$ large, since $-H(p)$ would be large and negative. In fact

$$\lim_{|p| \to \infty} \frac{pq - H(p)}{|p|} = -\infty$$

So

$$\lim_{|p| \to \infty} pq - H(p) = -\infty$$

Hence, the max exists!

If H satisfies $(*)$ then H^* does as well. Indeed

$$H^*(q) = \max_{p \in \mathbb{R}} \{ pq - H(p) \} \qquad q \neq 0$$

$$\geq \lambda \frac{q}{|q|} \cdot q - H\left(\lambda \frac{q}{|q|}\right) \qquad \text{choosing } p = \lambda \frac{q}{|q|}$$

$$= \lambda |q| - \max_{|p| \leq \lambda} H(p) \qquad \lambda > 0$$

Hence

$$\frac{H^*(q)}{|q|} \geq \lambda - \max_{|p| \leq \lambda} H(p)$$

and

$$\lim_{|q| \to \infty} \frac{H^*(q)}{|q|} \geq \lambda$$

Since this holds for all λ, $(*)$ holds for H^*.

Furthermore, H^* is convex, we will leave the proof of this to homework.

Our aim now is to show that the Hopf-Lax formula

$$u(x,t) = \min_{y \in \mathbb{R}} \left\{ f(y) + t\,L\left(\frac{x-y}{t}\right) \right\}$$

solves the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(u_x) = 0, & -\infty < x < \infty \\ & t > 0 \\ u(x,0) = f(x) & \end{cases}$$

where $L = H^*$. We assume

$$\boxed{H \text{ convex} \quad \text{and} \quad \lim_{|p| \to \infty} \frac{H(p)}{|p|} = \infty}$$

Proof next time....