Appendix to notes:

Recall the Hopf-Lax formula

$$u(x,t) = \min_{y \in \mathbb{R}} \left\{ f(y) + tL\left(\frac{x-y}{t}\right) \right\} \qquad \text{(H-L)}$$

We show here that

$$|u(x,t) - f(x)| \leq Ct$$

for a constant $C > 0$. This shows in particular that

$$\lim_{t \to 0^+} u(x,t) = f(x).$$

Proof: Take $y = x$ in (H-L) to get

$$u(x,t) \leq f(x) + tL(0)$$

or

$$u(x,t) - f(x) \leq Ct \quad , \quad C = L(0).$$

To show $u(x,t) - f(x) \geq -Ct$

we need to assume f is Lipschitz,

that is

$$(L) \qquad |f(x) - f(y)| \leq C|x-y|$$

for all $x, y \in \mathbb{R}$ and some constant $C>0$.

Then

$$u(x,t) = \min_{y\in\mathbb{R}} \left\{ f(y) + tL\left(\frac{x-y}{t}\right)\right\}$$

$$= \min_{y\in\mathbb{R}} \left\{ f(x) + f(y) - f(x) + tL\left(\frac{x-y}{t}\right)\right\}$$

by (L) $$\geq f(x) + \min_{y\in\mathbb{R}} \left\{ -C|x-y| + tL\left(\frac{x-y}{t}\right)\right\}$$

$\left(z = \frac{x-y}{t}\right)$ $$= f(x) + t\min_{z\in\mathbb{R}} \left\{ -C|z| + L(z)\right\}.$$

$$= f(x) - t\max_{z\in\mathbb{R}} \left\{ C|z| - L(z)\right\}$$

$$= f(x) - tH(C)$$

So $\quad u(x,t) - f(x) \geq -H(C)\,t \quad \blacksquare$