Math 5588

The Hopf-Lax formula

$$u(x,t) = \min_{y\in\mathbb{R}} \left\{ f(y) + tL\left(\frac{x-y}{t}\right) \right\}$$

for the solution of

$$(H) \quad \begin{cases} u_t + H(u_x) = 0, & -\infty < x < \infty \\ & t > 0 \\ u(x,0) = f(x) \end{cases}$$

is a special case of <u>optimal control theory</u>.

Optimal control theory is concerned with controlling a system (say, driving it from one state to another) while minimizing a cost (say, fuel, energy, etc.)

Applications include many fields of science and engineering, such as robotics.

The state of the system at time t is given by a vector $x(t) \in \mathbb{R}^n$, and the dynamics of the system evolve according to a nonlinear ODE

$$(*) \quad \begin{cases} \dot{x}(s) = f(x(s), \alpha(s)) & t < s < T \\ x(t) = x. \end{cases}$$

Here, $\dot{x}(s) = \frac{dx}{dt}(s)$, or can write $\frac{dx}{ds}$,

and $\alpha(s) =$ value of control at time s

The control can take values in some

set A, and

$$\mathcal{A} = \{ \alpha : [0,T] \to A \}$$

is the set of admissible controls.

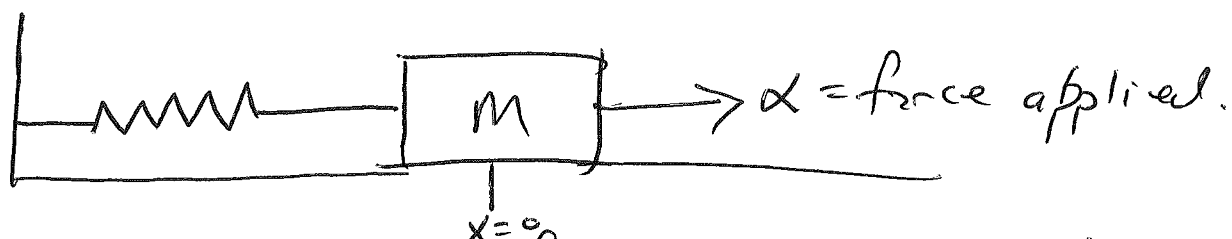
Each control $\alpha \in \mathcal{A}$ is associated with a cost

$$C_{x,t}(\alpha) := \int_t^T r(x(s), \alpha(s))\, ds + g(x(T))$$

The cost depends on the initial time t, the initial condition $x(t) = x$ of the system, and the control $\alpha \in \mathcal{A}$. The values of $x(s)$ for $t < s \le T$ are determined by solving $(*)$ and are of course influenced by the control α.

The function $r(x, \alpha)$ is called the <u>running cost</u> and $g(x)$ is called the <u>terminal cost</u>.

<u>Example</u>: Say we want to control a mass on a spring



Our control is a force $\alpha(s)$ that we can apply in the horizontal direction, and we can change the force over time. Other forces acting on the mass are the spring force following Hooke's law and <u>friction</u> between the mass and the floor.

Let $x=0$ be the relaxed position of the spring. By Hookes law the force of the spring is (to right)

$$F_{spring} = -kx$$

where $k>0$ is the spring constant. The force of friction is proportional to the force of gravity normal to the surface

$$F_{friction} = -\frac{\dot{x}(s)}{|\dot{x}(s)|}\mu mg$$

where μ = coefficient of friction, and

$$-\frac{\dot{x}(s)}{|\dot{x}(s)|} = -\text{sign}(\dot{x}(s))$$ ensures

that the force is in the opposite of the direction of motion. If $\dot{x}(s)=0$ we set $F=0$

By Newton's Law

$$\alpha(s) - kx(s) - \mu m g \frac{\dot{x}(s)}{|\dot{x}(s)|} = \underbrace{m\ddot{x}(s)}_{\text{mass} \times \text{acceleration}}$$

Set $x_1(s) = x(s)$ and $x_2(s) = \dot{x}_1(s) = \dot{x}(s)$

Then $\underline{x}(s) = (x_1(s), x_2(s))$ satisfies

$$\dot{x}_1(s) = x_2(s)$$

$$\dot{x}_2(s) = \frac{1}{m}\alpha(s) - \frac{k}{m}x_1(s) - \mu g \overbrace{\frac{\dot{x}(s)}{|\dot{x}(s)|}}^{\frac{x_2(s)}{|x_2(s)|}}$$

Or

$$\underbrace{\dot{\underline{x}}(s) = \begin{bmatrix} x_2(s) \\ \frac{1}{m}\alpha(s) - \frac{k}{m}x_1(s) - \mu g \frac{x_2(s)}{|x_2(s)|} \end{bmatrix}}_{f(\underline{x}(s), \alpha(s))}.$$

Say we have an initial position $x(0) = 0$.

The cost might be the total work done, or

$$C_{op}(\alpha) = \int_0^T \alpha(s)\,\dot{x}(s)\,ds$$

$$= \int_0^T \alpha(s)\,x_2(s)\,ds.$$

The problem of finding optimal controls is challenging. We will take a different approach: Instead of looking for optimal controls, we will try to compute the optimal <u>value</u> of $C(\alpha)$ over all controls. This gives the value function

$$\boxed{u(x,t) := \min_{\alpha \in A} C_{x,t}(\alpha)}$$

The value function $u: \mathbb{R}^n \times [0,T] \to \mathbb{R}$ gives the smallest value of the cost given the system starts at ~~p~~ state $x \in \mathbb{R}^n$ at time $t < T$.

$u(x,t)$ = Smallest value of cost given system starts at $X(t) = x$ at time t.

We will derive a PDE that $u(x,t)$ satisfies.

First, we prove a dynamic programming principle.

Theorem: For $h > 0$ small enough so that $0 < t+h < T$ we have

$$u(x,t) = \min_{\alpha \in A} \left\{ \int_t^{t+h} r(X(s), \alpha(s))\, ds + u(X(t+h), t+h) \right\}$$

where $X(s)$ solves

$$\begin{cases} \dot{x}(s) = f(x(s), \alpha(s)) & (t < s < t+h) \\ x(t) = x \end{cases}$$

<u>Proof:</u> For any control $\alpha_1 \in \mathcal{A}$ solve the ODE

$$\begin{cases} \dot{x}_1(s) = f(x_1(s), \alpha(s)) \\ x_1(t) = x. \end{cases}$$

Then choose an optimal control α_2 so that

$$u(x_1(t+h), t+h) = \int_{t+h}^{T} r(x_2(s), \alpha_2(s))\, ds + g(x_2(T))$$

where

$$\begin{cases} \dot{x}_2(s) = f(x_2(s), \alpha_2(s)) & (t+h < s < T) \\ x_2(t+h) = x_1(t+h) \end{cases}$$

Now concatenate the controls

$$\alpha_3(s) = \begin{cases} \alpha_1(s), & t \le s < t+h \\ \alpha_2(s), & t+h \le s \le T \end{cases}$$

and let X_3 solve

$$\begin{cases} \dot{x}_3(s) = f(x_3(s), \alpha_3(s)) \\ x_3(t) = x \end{cases}$$

By uniqueness of solutions of ODEs

$$x_3(s) = \begin{cases} x_1(s), & t \le s < t+h \\ x_2(s), & t+h \le s \le T \end{cases}$$

Hence

$$\begin{aligned} u(x,t) &\le C_{x,t}(\alpha_3) \\ &= \int_t^T r(x_3(s), \alpha_3(s))\, ds + g(x_3(T)) \\ &= \int_t^{t+h} r(x_1(s), \alpha_1(s))\, ds + \int_{t+h}^T r(x_2(s), \alpha_2(s))\, ds + g(x_2(T)) \\ &= \int_t^{t+h} r(x_1(s), \alpha_1(s))\, ds + u(x_1(t+h), t+h) \end{aligned}$$

Since α_1 is arbitrary

$$u(x,t) \le \min_{\alpha \in A} \left\{ \int_t^{t+h} r(x(s),\alpha(s))\, ds + u(x(t+h), t+h) \right\}.$$

For the other direction select α such that

$$u(x,t) = \int_t^T r(x(s),\alpha(s))\, ds + g(x(T))$$

when $x(s)$ solves for $t < s < t+h$ $\left\{ \begin{array}{l} \dot{x}(s) = f(x(s),\alpha(s)) \\ x(t) = x \end{array} \right.$

Then by definition

$$u(x(t+h), t+h) \le \int_{t+h}^T r(x(s),\alpha(s))\, ds + g(x(T))$$

and hence we have

$$u(x,t) \geq \int_t^{t+h} r(x(s), \alpha(s))\, ds + u(x(t+h), t+h).$$

It follows immediately that

$$u(x,t) \geq \min_{\alpha \in A} \left\{ \int_t^{t+h} r(x(s), \alpha(s))\, ds + u(x(t+h), t+h) \right\}.$$

$\blacksquare$

The dynamic programming principle

$$u(x,t) = \min_{\alpha \in A} \left\{ \int_t^{t+h} r(x(s), \alpha(s))\, ds + u(x(t+h), t+h) \right\}$$

allows us to derive a PDE that $u(x,t)$ satisfies called the Hamilton–Jacobi–Bellman equation.

Theorem: If $u(x,t)$ is smooth then u satisfies

$$\begin{cases} u_t + H(Du, x) = 0, & x \in \mathbb{R}^n, \ t < T \\ u(x,T) = g, & x \in \mathbb{R}^n \end{cases}$$

where $$H(p,x) = \min_{a \in A} \{ p \cdot f(x,a) + r(x,a) \}$$

Proof: Recall the Taylor expansion

$$u(y,s) = u(x,t) + Du(x,t) \cdot (y-x) + u_t(x,t)(s-t) + O(|x-y|^2 + |s-t|^2)$$

If $h>0$ is small then let us assume $\alpha(s) = a \in A$ is constant for $t \le s \le t+h$. Then $\dot{x}(s) = f(x(s), \alpha(s))$ is roughly constant for $t \le s \le t+h$ and

$$\dot{x}(s) \simeq f(x,a), \quad x(s) \simeq x.$$

Hence $$x(s) = x(t) + \int_t^s \dot{x}(\tau)\,d\tau$$

$$\approx x(t) + (s-t)\, f(x,a)$$

for $t \le s \le t+h$. This gives

$$x(t+h) \approx x + h f(x,a)$$

since $x(t) = x$. Therefore, by Taylor expansion

$$u(x(t+h), t+h) - u(x,t)$$

$$\approx Du(x,t) \cdot h f(x,a) + u_t(x,t)\, h.$$

Reorganizing the dynamic programming principle gives

$$\min_{a \in A} \left\{ u(x(t+h), t+h) - u(x,t) + \int_t^{t+h} r(x,a)\, ds \right\} \approx 0$$

Substituting the Taylor expansion gives

$$\min_{a \in A} \left\{ Du(x,t) \cdot h f(x,a) + u_t(x,t) h + \int_t^{t+h} r(x,a)\, ds \right\} = 0$$

Hence

$$u_t(x,t) + \min_{a \in A} \left\{ Du(x,t) \cdot f(x,a) + r(x,a) \right\} \simeq 0$$

This becomes exact as you send $h \to 0^+$ $\blacksquare$

After solving the Hamilton-Jacobi-Bellman equation, we can select the optimal control by choosing $a = \alpha(s)$ at time s to minimize

$$\min_{a \in A} \left\{ Du(x,t) \cdot f(x,a) + r(x,a) \right\}.$$

This is <u>feedback</u> <u>control</u>.

When u is not smooth we can still make the theorem hold provided we interpret solutions of the Hamilton-Jacobi-Bellman equation in the viscosity sense.

The general idea is to start from the dynamic programming principle

$$\min_{\alpha \in A} \left\{ u(x(t+h), t+h) - u(x,t) + \int_t^{t+h} r(x(s), \alpha(s))\, ds \right\} = 0$$

and let $u - \varphi$ have a local max at (x,t) when φ is smooth.

As for the Hopf-Lax formula we can assume

$$u(y,s) - \varphi(y,s) \leq u(x,t) - \varphi(x,t)$$

for all (y,s).

Hence

$$u(y,s) - u(x,t) \le \varphi(y,s) - \varphi(x,t)$$

for all (y,s). Plugging this into dynamic programming principle gives

$$\min_{\alpha \in A} \left\{ \varphi(x(t+h), t+h) - \varphi(x,t) + \int_t^{t+h} r(x(s),\alpha(s))\,ds \right\} \ge 0$$

Since φ is smooth we can use prev. theorem to get

$$\varphi_t(x,t) + H(D\varphi(x,t), x) \ge 0,$$

or

$$-\varphi_t(x,t) - H(D\varphi(x,t), x) \le 0$$

So u is a viscosity subsolution of

$$-\varphi_t(x,t) - H(\nabla\varphi(x,t), x) = 0$$

The supersolution property is similar and we find that $u(x,t)$ is a viscosity solution of

$$-\varphi_t - H(\nabla\varphi, x) = 0$$

The appearance of a minus sign is due to the terminal value problem (e.g., $u(x,T) = g(x)$, not initial value $u(x,0)$).