# Math 5588 | Differential games.

Recall a differential game involves two players with controls $\alpha(s)$, $\beta(s)$

$$\alpha : [0,T] \longrightarrow A \ , \quad \beta : [0,T] \longrightarrow B$$

and a dynamical system

$$\begin{cases} \dot{x}(s) = f(x(s), \alpha(s), \beta(s)) \ , & t < s \leq T \\ x(t) = x \end{cases}$$

That both players seek to control with diametrically opposing goals.

The <u>upper value</u> of the game is

$$\overline{u}(x,t) = \max_{S \in \Delta_2} \min_{\alpha \in A} C_{x,t}(\alpha, S(\alpha))$$

and the <u>lower</u> value is

$$\underline{u}(x,t) = \min_{S \in \Delta_1} \max_{\beta \in B} C_{x,t}(S(\beta), \beta)$$

where the cost is

$$C_{x,t}(\alpha,\beta) = \int_t^T r(x(s),\alpha(s),\beta(s))\,ds + g(x(T)),$$

and $\Delta_i$ is the set of non-anticipating strategies for player $i$ (see previous lecture for precise definitions).

We also have the dynamic programming principle

$$\bar{u}(x,t) = \max_{S\in\Delta_2}\min_{\alpha\in A}\left\{\int_t^{t+h} r(x(s),\alpha(s),\beta(s))\,ds + \bar{u}(x(t+h),t+h)\right\}$$

$$\beta = S[\alpha]$$

$$\underline{u}(x,t) = \min_{S\in\Delta_1}\max_{\beta\in B}\left\{\int_t^{t+h} r(x(s),\alpha(s),\beta(s))\,ds + \underline{u}(x(t+h),t+h)\right\}.$$

$$\alpha = S[\beta]$$

We can use the dynamic programming principle to show that $\bar{u}$ and $\underline{u}$ satisfy PDEs called Hamilton-Jacobi-Isaacs equations.

<u>Theorem</u> (Hamilton-Jacobi-Isaacs)

The upper value $\bar{u}$ satisfies the upper Isaacs equation

$$
(1)\quad \begin{cases} \bar{u}_t + \bar{H}(D\bar{u}, x) = 0, & 0 \le t < T,\ x \in \mathbb{R}^n \\ \bar{u}(x,T) = g(x), & x \in \mathbb{R}^n \end{cases}
$$

while the lower value $\underline{u}$ satisfies the lower Isaacs equation

$$
(2)\quad \begin{cases} \underline{u}_t + \underline{H}(D\underline{u}, x) = 0, & 0 \le t < T,\ x \in \mathbb{R}^n \\ \underline{u}(x,t) = g(x), & x \in \mathbb{R}^n \end{cases}
$$

where

$$
\bar{H}(p,x) = \min_{a \in A} \max_{b \in B} \{ f(x,a,b) \cdot p + r(x,a,b) \}
$$

and

$$
\underline{H}(p,x) = \max_{b \in B} \min_{a \in A} \{ f(x,a,b) \cdot p + r(x,a,b) \}.
$$

Remark: Notice the min and max switch between $\overline{u}$ and $\overline{H}$. This is important, and subtle.

Before proving Theorem, let us consider consequences.

Lemma: $\underline{H}(p,x) \le \overline{H}(p,x)$.

Proof: It is a general fact that max min $\le$ min max. To see why, by definition of $\overline{H}$ there exists $a^* \in A$ s.t.

$$\overline{H}(p,x) = \max_{b \in B} \{ f(x,a^*,b) \cdot p + r(x,a^*,b) \}$$

$$\ge \max_{b \in B} \min_{a \in A} \{ f(x,a,b) \cdot p + r(x,a,b) \}$$

$$= \underline{H}(p,x). \qquad \boxed{}$$

By the lemma

$$0 = \overline{u}_t + \overline{H}(\nabla \overline{u}, x) \geq \overline{u}_t + \underline{H}(\nabla \overline{u}, x),$$

that is,

$$(1) \begin{cases} \overline{u}_t + \underline{H}(\nabla \overline{u}, x) \leq 0, & 0 \leq t < T \\ & x \in \mathbb{R}^n \\ \overline{u}(x, T) = g(x), & \end{cases}$$

and

$$(2) \begin{cases} \underline{u}_t + \underline{H}(\nabla \underline{u}, x) = 0, & 0 \leq t < T, x \in \mathbb{R}^n \\ \underline{u}(x, T) = g(x). & \end{cases}$$

Property (1) is <u>actually</u> saying that $\overline{u}$ is a <u>supersolution</u> of the lower Isaacs equation. The inequality is reversed since this is a terminal value problem. Hence, we can show that

$$\underline{u} \le \bar{u}$$

via the maximum principle.

Sketch of proof: let $\varepsilon > 0$ and suppose

$$w(x,t) = \underline{u}(x,t) - \bar{u}(x,t) + \varepsilon(t-T)$$

is $\underbrace{\text{positive}}_{>0}$ somewhere, i.e., $\underline{u} \le \bar{u} - \varepsilon(t-T)$ does not hold. Let $(x_\varepsilon, t_\varepsilon) \in \mathbb{R}^n \times [0,T]$ be a point $w$ attains its positive max.

(We would need to do more work if max is not attained. There are just main ideas.)

Since $w(x,T) = 0$ and $w(x_\varepsilon, t_\varepsilon) > 0$

we have $0 \le t_\varepsilon < T$, hence

$$\nabla w(x_\varepsilon, t_\varepsilon) = 0 \quad \text{and} \quad w_t(x_\varepsilon, t_\varepsilon) \le 0$$

since max can be at $t_\varepsilon = 0$

Hence $\nabla \bar{u}(x_\varepsilon, t_\varepsilon) = \nabla \underline{u}(x_\varepsilon, t_\varepsilon)$

and $\underline{u}_t(x_\varepsilon, t_\varepsilon) \le \bar{u}_t(x_\varepsilon, t_\varepsilon) - \varepsilon$

Thus

$$\begin{aligned} 0 &= \underline{u}_t(x_\varepsilon, t_\varepsilon) + \underline{H}(x_\varepsilon, \nabla \underline{u}(x_\varepsilon, t_\varepsilon)) \\ &\le \bar{u}_t(x_\varepsilon, t_\varepsilon) - \varepsilon + \underline{H}(x_\varepsilon, \nabla \bar{u}(x_\varepsilon, t_\varepsilon)) \\ &\le \bar{u}_t(x_\varepsilon, t_\varepsilon) + \bar{H}(x_\varepsilon, \nabla \bar{u}(x_\varepsilon, t_\varepsilon)) - \varepsilon \\ &\le -\varepsilon \end{aligned}$$

, which is a contradiction.

Hence $\underline{u} \le \bar{u} + \varepsilon(T-t)$ for all $\varepsilon > 0$.

This establishes that the lower value is indeed less than the upper value, which justifies the name.

Corollary: If

$$\overline{H}(p,x) = \underline{H}(p,x), \text{ i.e.,}$$

$$\min_{a\in A} \max_{b\in B} \{f(x,a,b)\cdot p + r(x,a,b)\}$$

$$= \max_{b\in B} \min_{a\in A} \{f(x,a,b)\cdot p + r(x,a,b)\}$$

then $\overline{u} = \underline{u} = u$ and the game has a value.

Proof: $\overline{u} = \underline{u}$ by uniqueness of solutions of Hamilton-Jacobi equations (via maximum principle).

# Back to the Isaac's equation

We will prove that

$$(*) \quad \boxed{\bar{u}_t + \bar{H}(D\bar{u}, x) \geq 0}$$

The other inequality and proof for $\underline{u}$ are similar and are left as an exercise.

Recall:

$$\bar{H}(p, x) = \min_{a \in A} \max_{b \in B} \{ f(x,a,b) \cdot p + r(x,a,b) \}$$

and

$$\bar{u}(x,t) = \max_{S \in \Delta_2} \min_{\alpha \in A} \left\{ \int_t^{t+h} r(x(s), \alpha(s), S[\alpha](s))\,ds + u(x(t+h), t+h) \right\}$$

where

$$\begin{cases} \dot{x}(s) = f(x(s), \alpha(s), S[\alpha](s)), & t < s < t+h \\ x(t) = x \end{cases}$$

To prove $(*)$, assume to the contrary that

$$(**) \quad \bar{u}_t(x,t) + \bar{H}(D\bar{u}(x,t), x) = -\theta < 0$$

for some $\theta > 0$ and $(x,t) \in \mathbb{R}^n \times (-\infty, T)$.

This means

$$\bar{u}_t(x,t) + \min_{a \in A} \max_{b \in B} \left\{ f(x,a,b) \cdot \nabla \bar{u}(x,t) + r(x,a,b) \right\} = -\theta$$

Hence there exists $a^* \in A$ such that

$$(\square) \quad \max_{b \in B} \left\{ \bar{u}_t(x,t) + f(x,a^*,b) \cdot \nabla \bar{u}(x,t) + r(x,a^*,b) \right\} = -\theta$$

Let $\alpha(s) = a^*$ for $t < s < t+h$ and let $S \in \Delta_2$ be any non-anticipating strategy for player 2.

Let $x(s)$, $t < s < t+h$ solve

$$\begin{cases} \dot{x}(s) = f(x(s), \alpha(s), S[\alpha](s)), & t < s < t+h \\ x(t) = x \end{cases}$$

<u>Fact</u> $x(s)$ is continuous, even if $S[\alpha]$ is not.

By (□) if $h>0$ is sufficiently small we can ensure that

$$\max_{s\in\Delta_2}\left\{\bar{u}_t(x(s),s)+f(x(s),\alpha(s),S[\alpha](s))\cdot\nabla\bar{u}(x(s),s)+r(x(s),\alpha(s),S[\alpha](s))\right\}$$

$$\leq -\frac{\theta}{2}.$$

Since $\frac{d}{ds}u(x(s),s)=\bar{u}_t(x(s),s)+f(x(s),\alpha(s),S[\alpha](s))\cdot\nabla\bar{u}$

we have

$$\max_{S\in\Delta_2}\left\{\frac{d}{ds}\bar{u}(x(s),s)+r(x(s),\alpha(s),S[\alpha](s))\right\}\leq-\frac{\theta}{2}$$

Integrating both sides $s=t$ to $s=t+h$

$$\max_{S\in\Delta_2}\left\{\bar{u}(x(t+h),t+h)-\bar{u}(x,t)+\int_t^{t+h}r(x(s),\alpha(s),S[\alpha](s))\,ds\right\}$$

$$\leq-\frac{\theta}{2}$$

Rearranging, for $\alpha(s) = a^*$, $t < s < t+h$

$$\bar{u}(x,t) \geq \max_{S \in \Delta_2} \left\{ \int_t^{t+h} r(x(s), \alpha(s), S[\alpha](s))\,ds + \bar{u}(x(t+h), t+h) \right\} + \frac{\theta}{2}$$

This implies

$$\bar{u}(x,t) \geq \max_{S \in \Delta_2} \min_{\alpha \in A} \left\{ \int_t^{t+h} r(x(s), \alpha(s), S[\alpha](s))\,ds + \bar{u}(x(t+h), t+h) \right\} + \frac{\theta}{2}$$

which contradicts the dynamic programming principle as $\theta > 0$. $\blacksquare$

In general $\bar{u}$ and $\underline{u}$ are not differentiable, but all arguments can be made rigorous in viscosity sense.