Lecture 8 | Feb 9, 2017

# The Fourier Transform

Recall the complex form of the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{\ell}}, \qquad c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(y) e^{-i\frac{n\pi y}{\ell}}\, dy$$

for a function $f : [-\ell, \ell] \longrightarrow \mathbb{R}$.

We now send $\ell \longrightarrow \infty$. Write

$$K = \frac{n\pi}{\ell} \quad \text{and} \quad \Delta K = \frac{\pi}{\ell}$$

and substitute $c_n$ into Fourier series for $f$:

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2\ell} \int_{-\ell}^{\ell} f(y) e^{-i\frac{n\pi y}{\ell}}\, dy\, e^{i\frac{n\pi x}{\ell}}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left( \int_{-\ell}^{\ell} f(y) e^{-iKy}\, dy \right) e^{iKx} \Delta K$$

This is a Riemann sum, and as we send $\ell \longrightarrow \infty$ we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(y) e^{-iky} \, dy \right) e^{ikx} \, dk \quad \longleftarrow dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iky} \, dy \right) e^{ikx} \, dk.$$

We define the Fourier transform of $f$, denoted $\hat{f}$, by

$$\boxed{\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx}$$

The inverse Fourier transform is

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dk}$$

We also write $\hat{f} = \mathcal{F}(f)$

and $f = \mathcal{F}^{-1}(\hat{f})$

$\mathcal{F}$ = Fourier transform

$\mathcal{F}^{-1}$ = Inverse fourier transform

Properties of Fourier Transform:

(1) Linearity: $\mathcal{F}(f+g) = \mathcal{F}(f) + \mathcal{F}(g)$

and $\mathcal{F}(\alpha f) = \alpha \mathcal{F}(f)$

(2) Shifts: The Fourier transform of $f(x-a)$ is

$$\mathcal{F}(f(x-a)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a)\, e^{-ikx}\, dx$$

$y = x - a$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-ik(y+a)}\, dy$$

$$= e^{-ika} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-iky}\, dy$$

$$= e^{-ika}\, \hat{f}(k)$$

(3) Scaling: The Fourier transform of $f(ax)$ is

$$\mathcal{F}(f(ax)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{-ikx}\, dx$$

$y = ax$

$dy = |a| dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-i\left(\frac{k}{a}\right)y} \frac{dy}{|a|}$$

$$= \frac{1}{|a|} \hat{f}\left(\frac{k}{a}\right)$$

(4) Derivatives: The Fourier transform of $f'(x)$ is

$$\mathcal{F}(f'(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)\, e^{-ikx}\, dx$$

Integration By Parts

$$= \underbrace{\frac{1}{\sqrt{2\pi}} f(x)\, e^{-ikx} \Big|_{-\infty}^{\infty}}_{=0} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ik) f(x)\, e^{-ikx}\, dx$$

$\left(\text{if } \lim_{x \to \pm\infty} f(x) = 0\right)$

$$= ik\, \hat{f}(k)$$

So $\mathcal{F}(f'(x)) = ik\hat{f}(k)$

The Fourier transform turns differentiation into multiplication by frequency variable $k$.

$$\boxed{\text{Fourier transform turns \underline{differential equations} into \underline{algebraic equations} which are easier to solve!}}$$

Example: If $f'(x) + f(x) = g(x)$

then $ik\hat{f}(k) + \hat{f}(k) = \hat{g}(k)$

and $$\hat{f}(k) = \frac{\hat{g}(k)}{ik+1}$$

Easy to solve for $\hat{f}(k)$, but now we have to invert Fourier transform to find $f(x)$.

## ⑤ Higher derivatives

Note: $\mathcal{F}(f''(x)) = ik\,\mathcal{F}(f'(x))$

$$= (ik)^2 \mathcal{F}(f(x))$$

$$= -k^2 \hat{f}(k)$$

$$\mathcal{F}(f'''(x)) = -k^2 \mathcal{F}(f'(x))$$

$$= (ik)^2 (ik) \hat{f}(k)$$

$$= (ik)^3 \hat{f}(k).$$

In general

$$\mathcal{F}(f^{(m)}(x)) = (ik)^m \hat{f}(k)$$

where $f^{(m)} =$ $m^{th}$ derivative of $f$.

Example: If $f'''(x) + 2f''(x) + f'(x) - f(x) = g(x)$

then

$$(ik)^3 \hat{f}(k) + 2(ik)^2 \hat{f}(k) + \hat{f}(k)(ik) - \hat{f}(k) = \hat{g}(k)$$

or

$$\hat{f}(k) = \frac{\hat{g}(k)}{-ik^3 - 2k^2 + ik - 1}$$

How do we invert this?

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Some important Fourier transforms

① Delta function: $$\hat{\delta}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^0 = \frac{1}{\sqrt{2\pi}}.$$

Therefore

$$\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\delta}(k) e^{ikx} \, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dx$$

[This should be interpreted in distributional sense, since $e^{ikx}$ is not integrable]

Note

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \, dx$$

[Change variables $x \rightsquigarrow -x$]

$$= \frac{1}{\sqrt{2\pi}} \mathcal{F}(1)$$

Therefore $\boxed{\mathcal{F}(1) = \sqrt{2\pi}\, \delta(x)}$

2. By the shift property [cos and sin]

$$\mathcal{F}(\delta(x-a)) = e^{-ika}\hat{\delta}(k) = \frac{e^{-ika}}{\sqrt{2\pi}}$$

and $$\mathcal{F}(\delta(x+a)) = \frac{e^{ika}}{\sqrt{2\pi}}$$

<u>Hence</u>

$$\mathcal{F}\left(\delta(x+a) + \delta(x-a)\right) = \frac{2\cos(ka)}{\sqrt{2\pi}}$$

$$\mathcal{F}\left(\delta(x+a) - \delta(x-a)\right) = \frac{2i\sin(ka)}{\sqrt{2\pi}}$$

Therefore

$$\mathcal{F}^{-1}\left(\frac{\cos(ka)}{\sqrt{2\pi}}\right) = \frac{\delta(x+a) + \delta(x-a)}{2}$$

$$\mathcal{F}^{-1}\left(\frac{\sin(ka)}{\sqrt{2\pi}}\right) = \frac{\delta(x+a) - \delta(x-a)}{2i}$$

and so

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \cos(ka)\, e^{ikx}\, dk = \frac{\delta(x+a) + \delta(x-a)}{2}$$

Swapping $k$ and $x$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(ax)\,e^{ikx}\,dx = \frac{\delta(k+a)+\delta(k-a)}{2}$$

Now replace $k$ by $-k$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(ax)\,e^{-ikx}\,dx = \frac{\delta(-k+a)+\delta(-k-a)}{2}$$

Hence

$$\boxed{\mathcal{F}\left(\frac{\cos(ax)}{\sqrt{2\pi}}\right) = \frac{\delta(k+a)+\delta(k-a)}{2}}$$

since $\delta$ is even so $\delta(k)=\delta(-k)$.

Similarily

$$\boxed{\mathcal{F}\left(\frac{\sin(ax)}{\sqrt{2\pi}}\right) = \frac{\delta(k-a)-\delta(k+a)}{2i}}$$

③ <u>Gaussian</u> $f(x) = e^{-x^2/2}$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ikx} \, dx$$

(complete the square)
$$= \frac{1}{\sqrt{2\pi}} e^{-k^2/2} \int_{-\infty}^{\infty} e^{-(x+ik)^2/2} \, dx$$

Change variables $y = x + ik$
$$= \frac{1}{\sqrt{2\pi}} e^{-k^2/2} \int_{-\infty}^{\infty} e^{-y^2/2} \, dy$$

<u>Claim</u>: $\displaystyle\int_{-\infty}^{\infty} e^{-y^2/2} \, dy = \sqrt{2\pi}$ (proof on next page)

Therefore
$$\boxed{\hat{f}(k) = e^{-k^2/2}}$$

<u>Gaussian</u> is a fixed point of Fourier Transform.

$$\left(\int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right)^2 = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy$$

switch to polar coord

$$= \int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2/2}\,r\,dr\,d\theta$$

$$= 2\pi \int_0^{\infty} r e^{-r^2/2}\,dr$$

$$= -2\pi e^{-r^2/2}\Big|_0^{\infty}$$

$$= 2\pi$$

Hence

$$\int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \sqrt{2\pi}$$

By the scaling rule

$$\mathcal{F}\left(e^{-ax^2}\right) = \mathcal{F}\left(e^{-(\sqrt{2a}\,x)^2/2}\right)$$

$$= \frac{1}{\sqrt{2a}} \mathcal{F}\left(e^{-x^2/2}\right)\Big|_{k=\frac{k}{\sqrt{2a}}}$$

$$= \frac{1}{\sqrt{2a}} e^{-\left(\frac{k}{\sqrt{2a}}\right)^2/2}$$

$$= \frac{1}{\sqrt{2a}} e^{-k^2/4a}$$

$$\boxed{\mathcal{F}\left(e^{-ax^2}\right) = \frac{1}{\sqrt{2a}} e^{-k^2/4a}}$$