Lecture 9 | Math 5588 Feb 14, 2017

Example: The <u>fundamental solution</u> of the heat equation is defined as the solution of

$$\begin{cases} \Phi_t - \Phi_{xx} = 0, & t>0, \ -\infty < x < \infty \\ \Phi(x,0) = \delta(x) \end{cases}$$

Take the Fourier transform of $\Phi = \Phi(x,t)$ in x

$$\hat{\Phi}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(x,t) e^{-ikx} \, dx$$

Note:

$$\frac{d}{dt} \hat{\Phi}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_t(x,t) e^{-ikx} \, dx$$

$$= F(\Phi_t) = \hat{\Phi}_t$$

Since $\hat{\delta}(x) = \frac{1}{\sqrt{2\pi}}$ we have $\left(F(\Phi_{xx}) = -k^2 \hat{\Phi}(k,t)\right)$

$$\begin{cases} \frac{d}{dt} \hat{\Phi}(k,t) = -k^2 \hat{\Phi}(k,t) \\ \hat{\Phi}(k,0) = \frac{1}{\sqrt{2\pi}} \end{cases}$$

This is an ODE. Solution is

$$\hat{\Phi}(k,t) = \frac{1}{\sqrt{2\pi}} e^{-k^2 t}$$

<u>Recall</u>: $\mathcal{F}(e^{-x^2/2}) = e^{-k^2/2}$

and $\mathcal{F}(f(\tfrac{x}{a})) = a\hat{f}(ak)$ for $a>0$.

Set $a=\sqrt{2t}$. Then

$$\mathcal{F}\left(e^{-\left(\frac{x}{\sqrt{2t}}\right)^2/2}\right) = \sqrt{2t}\, e^{-k^2 t}$$

<u>Hence</u>

$$\mathcal{F}\left(\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\right) = e^{-k^2 t}$$

and so

$$\boxed{\Phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}}$$

Convolution: The convolution of $f(x)$ and $g(x)$ is defined by

$$(f*g)(x) = \cancel{\int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy}.$$

$$= \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

Note: $f*g = g*f$.

The Fourier transform of $f*g$ is

$$F(f*g)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f*g)(x)\, e^{-jkx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)\, g(x-y)\, e^{-jkx}\, dy\, dx$$

$z = x - y$
$dz = dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)\, g(z)\, e^{-jkz}\, e^{-jky}\, dy\, dz$$

$$= \sqrt{2\pi}\, \hat{f}(k)\, \hat{g}(k).$$

Hence

$$\boxed{\mathcal{F}(f*g)(k) = \sqrt{2\pi}\,\hat{f}(k)\,\hat{g}(k)}$$

In other words

$$\boxed{\frac{1}{\sqrt{2\pi}} f*g = \mathcal{F}^{-1}\left(\hat{f}(k)\,\hat{g}(k)\right)}$$

Back to heat equation: Let $u(x,t)$ solve

$$(*)\begin{cases} u_t - u_{xx} = 0, & t>0,\ -\infty<x<\infty \\ u(x,0) = f(x) \end{cases}$$

Take Fourier transform of u in x

$$\hat{u}(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\,e^{-ikx}\,dx$$

Then

$$\begin{cases} \frac{d}{dt}\hat{u}(k,t) = -k^2\hat{u}(k,t), & t>0 \\ \hat{u}(k,0) = \hat{f}(k) \end{cases}$$

Therefore

$$\hat{u}(k,t) = \hat{f}(k) e^{-k^2 t}$$

and so

$$u(x,t) = \mathcal{F}^{-1}\left(\hat{f}(k) e^{-k^2 t}\right)$$

$$= \sqrt{2\pi}\, \mathcal{F}^{-1}\left(\hat{f}(k) \frac{1}{\sqrt{2\pi}} e^{-k^2 t}\right)$$

$$= \sqrt{2\pi}\, \mathcal{F}^{-1}\left(\hat{f}(k)\, \hat{\Phi}(k,t)\right)$$

$$= (f * \Phi)(x,t) \qquad \left(\text{convolution in } x \text{ only}\right)$$

So the solution of $(*)$ is

$$u(x,t) = (f * \Phi)(x,t)$$

$$= \int_{-\infty}^{\infty} f(y)\, \Phi(x-y,t)\, dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(y)\, e^{-(x-y)^2/4t}\, dy.$$

The wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0 & t>0, \quad -\infty < x < \infty \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Take Fourier transform to get

$$\begin{cases} \frac{d^2}{dt^2}\hat{u}(k,t) = -k^2\hat{u}(k,t) \\ \hat{u}(k,0) = \hat{f}(k) \\ \frac{d}{dt}\hat{u}(k,t)\Big|_{t=0} = \hat{g}(k) \end{cases}$$

Solution is

$$\hat{u}(k,t) = A\cos(kt) + B\sin(kt)$$

$$\begin{cases} \hat{u}(k,0) = A = \hat{f}(k) \\ \frac{d}{dt}\Big|_{t=0} \hat{u}(k,t) = Bk = \hat{g}(k) \end{cases} \quad , \quad \text{hence}$$

$$\boxed{\hat{u}(k,t) = \hat{f}(k)\cos(kt) + \frac{\hat{g}(k)}{k}\sin(kt)}$$

Recall: $\mathcal{F}\left(\frac{\delta(x+t)+\delta(x-t)}{2}\right) = \frac{\cos(kt)}{\sqrt{2\pi}}$

$$\mathcal{F}\left(\frac{\delta(x+t)-\delta(x-t)}{2i}\right) = i\,\frac{\sin(kt)}{\sqrt{2\pi}}$$

let $\quad G(x) = \int_0^x g(s)\,ds$

Then $\quad G'(x) = g(x) \quad$ and

$$\hat{g}(k) = \mathcal{F}(G') = ik\,\hat{G}(k)$$

so $\quad \frac{\hat{g}(k)}{k} = i\,\hat{G}(k)$

Therefore

$$\hat{u}(k,t) = \hat{f}(k)\cos(kt) + \hat{G}(k)\,i\sin(kt)$$

and so

$$u(x,t) = f * \left(\frac{\delta(x+t)+\delta(x-t)}{2}\right) + G * \left(\frac{\delta(x+t)-\delta(x-t)}{2}\right)$$

Note :

$$f * \delta(x-t) = \int_{-\infty}^{\infty} f(y)\,\delta(x-t-y)\,dy = f(x-t)$$

$$f * \delta(x+t) = \int_{-\infty}^{\infty} f(y)\,\delta(x+t-y)\,dy = f(x+t)$$

Hence

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) + \frac{1}{2}\left(G(x+t) - G(x-t)\right)$$

This is d'Alembert's formula

$$\frac{1}{2}\left(G(x+t) - G(x-t)\right)$$

$$= \frac{1}{2}\int_0^{x+t} g(s)\,ds - \frac{1}{2}\int_0^{x-t} g(s)\,ds$$

$$= \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

Plancheral: Recall for complex-valued functions f, g:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx \qquad \begin{pmatrix} z = x + iy \\ \bar{z} = x - iy \end{pmatrix}$$

$$\|f\|^2 = \langle f, f \rangle = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

$\bar{z}$ = complex conjugate of z

$$L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} \; : \; \|f\| < \infty \}$$

$L^2(\mathbb{R})$ is a Hilbert space with inner-product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, dx.$$

Q: What is $\langle \hat{f}, \hat{g} \rangle$?

$$\langle \hat{f}, \hat{g} \rangle = \int_{-\infty}^{\infty} \hat{f}(k)\, \overline{\hat{g}(k)}\, dk$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx\, \overline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ikx}\, dx}\; dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\, \overline{g(y)}\, e^{-ikx} e^{iky}\, dx\, dy\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \overline{g(y)} \underbrace{\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik(x-y)} \, dk \right]}_{= \mathcal{F}(1)(x-y) \; = \sqrt{2\pi}\, \delta(x-y)} dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \overline{g(y)} \, \delta(x-y) \, dx\, dy$$

$$= \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx$$

Therefore $\boxed{\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle}$

The <u>Fourier</u> <u>transform</u> is an Hilbert space isomorphism

$$\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}).$$

<u>Taking $f = g$</u> $\boxed{\|\hat{f}\| = \|f\|}$