The Calculus of Variations

Jeff Calder

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1 Introduction

The calculus of variations is a field of mathematics concerned with minimizing (or maximizing) functionals (that is, real-valued functions whose inputs are functions). The calculus of variations has a wide range of applications in physics, engineering, applied and pure mathematics, and is intimately connected to partial differential equations (PDEs).

For example, a classical problem in the calculus of variations is finding the shortest path between two points. The notion of length need not be Euclidean, or the path may be constrained to lie on a surface, in which case the shortest path is called a *geodesic*. In physics, Hamilton's principle states that trajectories of a physical system are critical points of the action functional. Critical points may be minimums, maximums, or saddle points of the action functional. In computer vision, the problem of segmenting an image into meaningful regions is often cast as a problem of minimizing a functional over all possible segmentations—a natural problem in the calculus of variations. Likewise, in image processing, the problem of restoring a degraded or noisy images has been very successfully formulated as a problem in the calculus of variations.

PDEs arise as the necessary conditions for minimizers of functionals. Recall in multi-variable calculus that if a function $u : \mathbb{R}^n \to \mathbb{R}$ has a minimum at $x \in \mathbb{R}^n$ then $\nabla u(x) = 0$. The necessary condition $\nabla u(x) = 0$ can be used to solve for candidate minimizers x. In the calculus of variations, if a function $u : \mathbb{R}^n \to \mathbb{R}$ is a minimizer of a functional $I(u)$ then the necessary condition $\nabla I(u) = 0$ turns out to be a PDE called the *Euler-Lagrange equation*. Studying the Euler-Lagrange equation allows us to explicitly compute minimizers and to study their properties. For this reason, there is a rich interplay between the calculus of variations and the theory of PDEs.

These notes aim to give a brief overview of the calculus of variations at the advanced undergraduate level. We will only assume knowledge of multivariable calculus and will avoid real analysis where possible. It is a good idea to review the appendix for some mathematical preliminaries and notation.

1.1 Examples

We begin with a parade of examples.

Example 1 (Shortest path)**.** Let A and B be two points in the plane. What is the shortest path between A and B? The answer depends on how we measure length! Suppose the length of a short line segment near (x, y) is the usual Euclidean length multiplied by a positive scale factor $g(x, y)$. For example, the length of a path could correspond to the length of time it would take a robot to navigate the path, and certain regions in space may be easier or harder to navigate, yielding larger or smaller values of g. Robotic navigation is thus a special case of finding the shortest path between two points.

Suppose A lies to the left of B and the path is a graph $u(x)$ over the x axis. See Figure 1. Then the "length" of the path between x and $x + \Delta x$ is approximately

$$L = g(x, u(x))\sqrt{1 + u'(x)^2}\Delta x.$$

If we let $A = (0, 0)$ and $B = (a, b)$ where $a > 0$, then the length of a path $(x, u(x))$ connecting A to B is

$$I(u) = \int_0^a g(x, u(x))\sqrt{1 + u'(x)^2}\, dx.$$

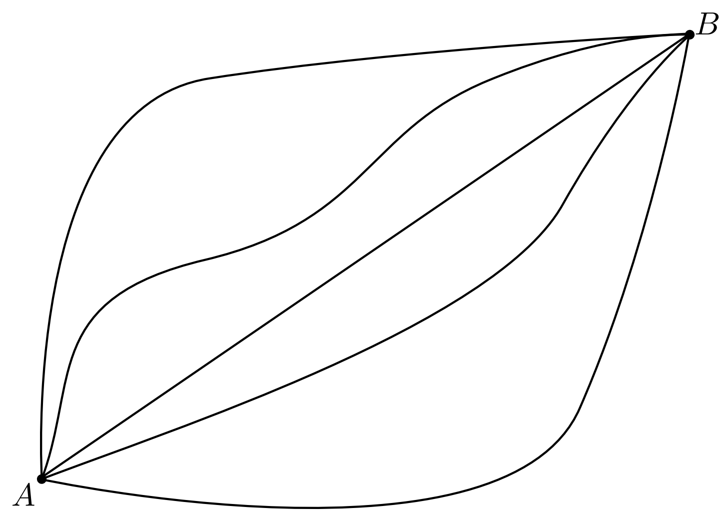


Figure 1: In our version of the shortest path problem, all paths must be graphs of functions $u = u(x)$.

The problem of finding the shortest path from A to B is equivalent to finding the function u that minimizes the functional $I(u)$ subject to $u(0) = 0$ and $u(a) = b$.

Example 2 (Brachistochrone problem)**.** In 1696 Johann Bernoulli posed the following problem. Let A and B be two points in the plane with A lying above B. Suppose we connect A and B with a thin wire and allow a bead to slide from A to B under the influence of gravity. Assuming the bead slides without friction, what is the shape of the wire that minimizes the travel time of the bead? Perhaps counter-intuitively, it turns out that the optimal shape is not a straight line! The problem is commonly referred to as the *brachistochrone problem*—the word brachistochrone derives from ancient Greek meaning “shortest time”.

Let g denote the acceleration due to gravity. Suppose that $A = (0, 0)$ and $B = (a, b)$ where $a > 0$ and $b < 0$. Let $u(x)$ for $0 \leq x \leq a$ describe the shape of the wire, so $u(0) = 0$ and $u(a) = b$. Let $v(x)$ denote the speed of the bead when it is at position x. When the bead is at position $(x, u(x))$ along the wire, the potential energy stored in the bead is $\mathrm{PE} = mgu(x)$ (relative to height zero), and the kinetic energy is $\mathrm{KE} = \frac{1}{2}mv(x)^2$, where m is the mass of the bead. By conservation of energy

$$\frac{1}{2}mv(x)^2 + mgu(x) = 0,$$

since the bead starts with zero total energy at point A. Therefore

$$v(x) = \sqrt{-2gu(x)}.$$

Between x and $x + \Delta x$, the bead slides a distance of approximately $\sqrt{1 + u'(x)^2}\Delta x$ with a speed of $v(x) = \sqrt{-2gu(x)}$. Hence it takes approximately

$$t = \frac{\sqrt{1 + u'(x)^2}}{\sqrt{-2gu(x)}}\Delta x$$

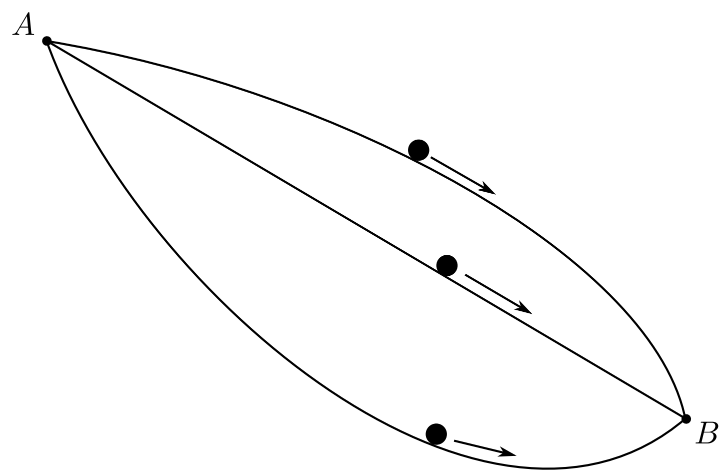


Figure 2: Depiction of possible paths for the brachistochrone problem.

time for the bead to move from position x to $x + \Delta x$. Therefore the total time taken for the bead to slide down the wire is given by

$$I(u) = \frac{1}{\sqrt{2g}} \int_0^a \sqrt{\frac{1 + u'(x)^2}{-u(x)}}\, dx.$$

The problem of determining the optimal shape of the wire is therefore equivalent to finding the function $u(x)$ that minimizes $I(u)$ subject to $u(0) = 0$ and $u(a) = b$.

Example 3 (Minimal surfaces)**.** Suppose we bend a piece of wire into a loop of any shape we wish, and then dip the wire loop into a solution of soapy water. A soap bubble will form across the loop, and we may naturally wonder what shape the bubble will take. Physics tells us that soap bubble formed will be the one with least surface area, at least locally, compared to all other surfaces that span the wire loop. Such a surface is called a *minimal surface*.

To formulate this mathematically, suppose the loop of wire is the graph of a function $g : \partial U \to \mathbb{R}$, where $U \subset \mathbb{R}^2$ is open and bounded. We also assume that all possible surfaces spanning the wire can be expressed as graphs of functions $u : \overline{U} \to \mathbb{R}$. To ensure the surface connects to the wire we ask that $u = g$ on ∂U. The surface area of a candidate soap film surface u is given by

$$I(u) = \int_U \sqrt{1 + |\nabla u|^2}\, dx.$$

Thus, the minimal surface problem is equivalent to finding a function u that minimizes I subject to $u = g$ on ∂U.

Example 4 (Image restoration)**.** A grayscale image is a function $u : [0,1]^2 \to [0,1]$. For $x \in \mathbb{R}^2$, $u(x)$ represents the brightness of the pixel at location x. In real-world applications, images are often corrupted in the acquisition process or thereafter, and we observe a noisy version of the image. The task of image restoration is to recover the true noise-free image from a noisy observation.

Let $f(x)$ be the observed noisy image. A widely used and very successful approach to image restoration is the so-called total variation (TV) restoration, which minimizes the functional

$$I(u) = \int_U \frac{1}{2}(u-f)^2 + \lambda|\nabla u|\,dx,$$

where $\lambda > 0$ is a parameter and $U = (0,1)^2$. The restored image is the function u that minimizes I (we do not impose boundary conditions on the minimizer). The first term $\frac{1}{2}(u-f)^2$ is a called a fidelity term, and encourages the restored image to be close to the observed noisy image f. The second term $|\nabla u|$ measures the amount of noise in the image and minimizing this term encourages the removal of noise in the restored image. The name TV restoration comes from the fact that $\int_U |\nabla u|\,dx$ is called the total variation of u. Total variation image restoration was pioneered by Rudin, Osher, and Fatemi [2].

Example 5 (Image segmentation)**.** A common task in computer vision is the segmentation of an image into meaningful regions. Let $f : [0,1]^2 \to [0,1]$ be a grayscale image we wish to segment. We represent possible segmentations of the image by the level sets of functions $u : [0,1]^2 \to \mathbb{R}$. Each function u divides the domain $[0,1]^2$ into two regions defined by

$$R^+(u) = \{x \in [0,1]^2 \,:\, u(x) > 0\} \quad \text{and} \quad R^-(u) = \{x \in [0,1]^2 \,:\, u(x) \le 0\}.$$

The boundary between the two regions is the level set $\{x \in [0,1]^2 \,:\, u(x) = 0\}$.

At a very basic level, we might assume our image is composed of two regions with different intensity levels $f = a$ and $f = b$, corrupted by noise. Thus, we might propose to segment the image by minimizing the functional

$$I(u,a,b) = \int_{R^+(u)} (f(x)-a)^2\,dx + \int_{R^-(u)} (f(x)-b)^2\,dx,$$

over all possible segmentations u and real numbers a and b. However, this turns out not to work very well since it does not incorporate the geometry of the region in any way. Intuitively, a semantically meaningful object in an image is usually concentrated in some region of the image, and might have a rather smooth boundary. The minimizers of I could be very pathological and oscillate rapidly trying to capture every pixel near a in one region and those near b in another region. For example, if f only takes the values 0 and 1, then minimizing I will try to put all the pixels in the image where f is 0 into one region, and all those where f is 1 into the other region, and will choose $a = 0$ and $b = 1$. This is true regardless of whether the region where f is zero is a nice circle in the center of the image, or if we randomly choose each pixel to be 0 or 1. In the later case, the segmentation u will oscillate wildly and does not give a meaningful result.

A common approach to fixing this issue is to include a penalty on the length of the boundary between the two regions. Let us denote the length of the boundary between $R^+(u)$ and $R^-(u)$ (i.e., the zero level set of u) by $L(u)$. Thus, we seek instead to minimize the functional

$$I(u,a,b) = \int_{R^+(u)} (f(x)-a)^2\,dx + \int_{R^-(u)} (f(x)-b)^2\,dx + \lambda L(u),$$

where $\lambda > 0$ is a parameter. Segmentation of an image is therefore reduced to finding a function $u(x)$ and real numbers a and b minimizing $I(u,a,b)$, which is a problem in the calculus of variations. This widely used approach was proposed by Chan and Vese in 2001 and is called Active Contours Without Edges [1].

The dependence of I on u is somewhat obscured in the form above. Let us write the functional in another way. Recall the Heaviside function H is defined as

$$H(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0. \end{cases}$$

Then the region $R^+(u)$ is precisely the region where $H(u(x)) = 1$, and the region $R^-(u)$ is precisely where $H(u(x)) = 0$. Therefore

$$\int_{R^+(u)} (f(x) - a)^2 \, dx = \int_U H(u(x))(f(x) - a)^2 \, dx,$$

where $U = (0, 1)^2$. Likewise

$$\int_{R^-(u)} (f(x) - b)^2 \, dx = \int_U (1 - H(u(x))) \, (f(x) - b)^2 \, dx.$$

We also have the identity (see Section A.10)

$$L(u) = \int_U |\nabla H(u(x))| \, dx = \int_U \delta(u(x)) |\nabla u(x)| \, dx.$$

Therefore we have

$$I(u, a, b) = \int_U H(u)(f - a)^2 + (1 - H(u)) \, (f - b)^2 + \lambda \delta(u) |\nabla u| \, dx.$$

2 The Euler-Lagrange equation

We aim to study general functionals of the form

$$I(u) = \int_U L(x, u(x), \nabla u(x)) \, dx, \tag{2.1}$$

where $U \subset \mathbb{R}^n$ is open and bounded, and L is a function

$$L : U \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}.$$

The function L is called the *Lagrangian*. We will write $L = L(x, z, p)$ where $x \in U$, $z \in \mathbb{R}$ and $p \in \mathbb{R}^n$. Thus, z represents the variable where we substitute $u(x)$, and p is the variable where we substitute $\nabla u(x)$. Writing this out completely we have

$$L = L(x_1, x_2, \ldots, x_n, z, p_1, p_2, \ldots, p_n).$$

The partial derivatives of L will be denoted $L_z(x, z, p)$,

$$\nabla_x L(x, z, p) = (L_{x_1}(x, z, p), \ldots, L_{x_n}(x, z, p)),$$

and

$$\nabla_p L(x, z, p) = (L_{p_1}(x, z, p), \ldots, L_{p_n}(x, z, p)).$$

Each example from Section 1.1 involved a functional of the general form of (2.1). For the shortest path problem $n = 1$ and

$$L(x, z, p) = g(x_1, z)\sqrt{1 + p_1^2}.$$

For the brachistochrone problem $n = 1$ and

$$L(x, z, p) = \sqrt{\frac{1 + p_1^2}{-z}}.$$

For the minimal surface problem $n = 2$ and

$$L(x, z, p) = \sqrt{1 + |p|^2}.$$

For the image restoration problem $n = 2$ and

$$L(x, z, p) = \frac{1}{2}(z - f(x))^2 + \lambda|p|.$$

Finally, for the image segmentation problem

$$L(x, z, p) = H(z)(f(x) - a)^2 + (1 - H(z))\,(f(x) - b)^2 + \lambda\delta(z)|p|.$$

We will always assume that L is smooth, and the boundary condition $g : \partial U \to \mathbb{R}$ is smooth. We now give necessary conditions for minimizers of I.

Theorem 1 (Euler-Lagrange equation)**.** *Suppose that $u \in C^2(\overline{U})$ satisfies*

$$I(u) \leq I(v) \tag{2.2}$$

for all $v \in C^2(\overline{U})$ with $v = u$ on ∂U. Then

$$L_z(x, u, \nabla u) - \operatorname{div}(\nabla_p L(x, u, \nabla u)) = 0 \quad \text{in } U. \tag{2.3}$$

Proof. Let $\varphi \in C_c^\infty(U)$ and set $v = u + t\varphi$ for a real number t. Since $\varphi = 0$ on ∂U we have $u = v$ on ∂U. Thus, by assumption

$$I(u) \leq I(v) = I(u + t\varphi) \quad \text{for all } t \in \mathbb{R}.$$

This means that $h(t) := I(u + t\varphi)$ has a global minimum at $t = 0$, i.e., $h(0) \leq h(t)$ for all t. It follows that $h'(t) = 0$, which is equivalent to

$$\frac{d}{dt}\Big|_{t=0} I(u + t\varphi) = 0. \tag{2.4}$$

We now compute the derivative in (2.4). Notice that

$$I(u + t\varphi) = \int_U L(x, u(x) + t\varphi(x), \nabla u(x) + t\nabla\varphi(x))\, dx.$$

For notational simplicity, let us suppress the x arguments from $u(x)$ and $\varphi(x)$. By the chain rule

$$\frac{d}{dt}L(x, u + t\varphi, \nabla u + t\nabla\varphi) = L_z(x, u + t\varphi, \nabla u + t\nabla\varphi)\varphi + \nabla_p L(x, u + t\varphi, \nabla u + t\nabla\varphi) \cdot \nabla\varphi.$$

Therefore

$$\left.\frac{d}{dt}\right|_{t=0} L(x, u + t\varphi, \nabla u + t\nabla\varphi) = L_z(x, u, \nabla u)\varphi + \nabla_p L(x, u, \nabla u) \cdot \nabla\varphi,$$

and we have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} I(u + t\varphi) &= \left.\frac{d}{dt}\right|_{t=0} \int_U L(x, u(x) + t\varphi(x), \nabla u(x) + t\nabla\varphi(x))\, dx \\
&= \int_U \left.\frac{d}{dt}\right|_{t=0} L(x, u(x) + t\varphi(x), \nabla u(x) + t\nabla\varphi(x))\, dx \\
&= \int_U L_z(x, u, \nabla u)\varphi + \nabla_p L(x, u, \nabla u) \cdot \nabla\varphi\, dx \\
&= \int_U L_z(x, u, \nabla u)\varphi\, dx + \int_U \nabla_p L(x, u, \nabla u) \cdot \nabla\varphi\, dx. \qquad (2.5)
\end{aligned}$$

Since $\varphi = 0$ on ∂U we can use the Divergence Theorem (Theorem 8) to compute

$$\int_U \nabla_p L(x, u, \nabla u) \cdot \nabla\varphi\, dx = -\int_U \operatorname{div}(\nabla_p L(x, u, \nabla u))\, \varphi\, dx.$$

Combining this with (2.4) and (2.5) we have

$$0 = \left.\frac{d}{dt}\right|_{t=0} I(u + t\varphi) = \int_U \big(L_z(x, u, \nabla u) - \operatorname{div}(\nabla_p L(x, u, \nabla u))\big)\varphi\, dx.$$

It follows from the vanishing lemma (Lemma 2 in the appendix) that

$$L_z(x, u, \nabla u) - \operatorname{div}(\nabla_p L(x, u, \nabla u)) = 0$$

everywhere in U, which completes the proof. □

Remark 1. Theorem 1 says that minimizers of the functional I satisfy the PDE (2.3). The PDE (2.3) is called the *Euler-Lagrange* equation for I.

Definition 1. A solution u of the Euler-Lagrange equation (2.3) is called a *critical point* of I.

Remark 2. In dimension $n = 1$ we write $x = x_1$ and $p = p_1$. Then the Euler-Lagrange equation is

$$L_z(x, u(x), u'(x)) - \frac{d}{dx}L_p(x, u(x), u'(x)) = 0.$$

Remark 3. In the proof of Theorem 1 we showed that

$$\int_U L_z(x, u, \nabla u)\varphi + \nabla_p L(x, u, \nabla u) \cdot \nabla\varphi\, dx = 0$$

for all $\varphi \in C_c^\infty(U)$. A function $u \in C^1(U)$ satisfying the above for all $\varphi \in C_c^\infty(U)$ is called a *weak* solution of the Euler-Lagrange equation (2.3). Thus, weak solutions of PDEs arise naturally in the calculus of variations.

Example 6. Consider the problem of minimizing the Dirichlet energy

$$I(u) = \int_U \frac{1}{2}|\nabla u|^2 - uf\,dx, \tag{2.6}$$

over all u satisfying $u = g$ on ∂U. Here, $f : U \to \mathbb{R}$ and $g : \partial U \to \mathbb{R}$ are given functions, and

$$L(x, z, p) = \frac{1}{2}|p|^2 - zf(x).$$

Therefore

$$L_z(x, z, p) = -f(x) \quad \text{and} \quad \nabla_p L(x, z, p) = p,$$

and the Euler-Lagrange equation is

$$-f(x) - \operatorname{div}(\nabla u) = 0 \ \text{ in } U.$$

This is Poisson's equation

$$-\Delta u = f \ \text{ in } U$$

subject to the boundary condition $u = g$ on ∂U.

Exercise 1. Derive the Euler-Lagrange equation for the problem of minimizing

$$I(u) = \int_U \frac{1}{p}|\nabla u|^p - uf\,dx$$

subject to $u = g$ on ∂U, where $p \geq 1$.

Example 7. The Euler-Lagrange equation in dimension $n = 1$ can be simplified when L has no x-dependence, so $L = L(z, p)$. In this case the Euler-Lagrange equation reads

$$L_z(u(x), u'(x)) = \frac{d}{dx}L_p(u(x), u'(x)).$$

Using the Euler-Lagrange equation and the chain rule we compute

$$\begin{aligned}
\frac{d}{dx}L(u(x), u'(x)) &= L_z(u(x), u'(x))u'(x) + L_p(u(x), u'(x))u''(x) \\
&= u'(x)\frac{d}{dx}L_p(u(x), u'(x)) + L_p(u(x), u'(x))u''(x) \\
&= \frac{d}{dx}\left(u'(x)L_p(u(x), u'(x))\right).
\end{aligned}$$

Therefore

$$\frac{d}{dx}\left(L(u(x), u'(x)) - u'(x)L_p(u(x), u'(x))\right) = 0.$$

It follows that

$$L(u(x), u'(x)) - u'(x)L_p(u(x), u'(x)) = C \tag{2.7}$$

for some constant C. This form of the Euler-Lagrange equation is often easier to solve when L has no x-dependence.

In some of the examples presented in Section 1.1, such as the image segmentation and restoration problems, we did not impose any boundary condition on the minimizer u. For such problems, Theorem 1 still applies, but the Euler-Lagrange equation (2.3) is not uniquely solvable without a boundary condition. Hence, we need some additional information about minimizers in order for the Euler-Lagrange equation to be useful for these problems.

Theorem 2. *Suppose that $u \in C^2(\overline{U})$ satisfies*

$$I(u) \leq I(v) \tag{2.8}$$

for all $v \in C^2(\overline{U})$. Then u satisfies the Euler-Lagrange equation (2.3) with boundary condition

$$\nabla_p L(x, u, \nabla u) \cdot \nu = 0 \quad \text{on} \quad \partial U. \tag{2.9}$$

Proof. By Theorem 1, u satisfies the Euler-Lagrange equation (2.3). We just need to show that u also satisfies the boundary condition (2.9).

Let $\varphi \in C^\infty(\overline{U})$ be a smooth function that is not necessarily zero on ∂U. Then by hypothesis $I(u) \leq I(u + t\varphi)$ for all t. Therefore, as in the proof of Theorem 1 we have

$$0 = \frac{d}{dt}\Big|_{t=0} I(u + t\varphi) = \int_U L_z(x, u, \nabla u)\varphi + \nabla_p L(x, u, \nabla u) \cdot \nabla\varphi \, dx. \tag{2.10}$$

Applying the Divergence Theorem (Theorem 8) we find that

$$0 = \int_{\partial U} \varphi \, \nabla_p L(x, u, \nabla u) \cdot \nu \, dS + \int_U \left(L_z(x, u, \nabla u) - \operatorname{div}\left(\nabla_p L(x, u, \nabla u)\right)\right) \varphi \, dx.$$

Since u satisfies the Euler-Lagrange equation (2.3), the second term above vanishes and we have

$$\int_{\partial U} \varphi \, \nabla_p L(x, u, \nabla u) \cdot \nu \, dS = 0$$

for all test functions $\varphi \in C^\infty(\overline{U})$. By a slightly different version of the vanishing lemma (Lemma 2 in the appendix) we have that

$$\nabla_p L(x, u, \nabla u) \cdot \nu = 0 \quad \text{on} \quad \partial U.$$

This completes the proof. □

2.1 The gradient interpretation

We can interpret the Euler-Lagrange equation (2.3) as the gradient of I. That is, in a certain sense (2.3) is equivalent to $\nabla I(u) = 0$.

To see why, let us consider a function $u : \mathbb{R}^n \to \mathbb{R}$. The gradient of u is defined as the vector of partial derivatives

$$\nabla u(x) = (u_{x_1}(x), u_{x_2}(x), \ldots, u_{x_n}(x)).$$

By the chain rule we have

$$\frac{d}{dt}\Big|_{t=0} u(x + tv) = \nabla u(x) \cdot v \tag{2.11}$$

for any vector $v \in \mathbb{R}^n$. It is possible to take (2.11) as the *definition* of the gradient of u. By this, we mean that $w = \nabla u(x)$ is the unique vector satisfying

$$\frac{d}{dt}\Big|_{t=0} u(x + tv) = w \cdot v$$

for all $v \in \mathbb{R}^n$.

In the case of functionals $I(u)$, we showed in the proof of Theorem 1 that

$$\frac{d}{dt}\Big|_{t=0} I(u + t\varphi) = \langle L_z(x, u, \nabla u) - \operatorname{div}\left(\nabla_p L(x, u, \nabla u)\right), \varphi\rangle_{L^2(U)}$$

for all $\varphi \in C_c^\infty(U)$. Here, the L^2-inner product plays the role of the dot product from the finite dimensional case. Thus, it makes sense to define the *gradient*, also called the *functional gradient* to be

$$\nabla I(u) := L_z(x, u, \nabla u) - \operatorname{div}\left(\nabla_p L(x, u, \nabla u)\right), \tag{2.12}$$

so that we have the identity

$$\frac{d}{dt}\Big|_{t=0} I(u + t\varphi) = \langle \nabla I(u), \varphi\rangle_{L^2(U)}. \tag{2.13}$$

The reader should compare this with the ordinary chain rule (2.11). Notice the definition of the gradient ∇I depends on the choice of the L^2-inner product. Using other inner products will result in different notions of gradient.

To numerically compute solutions of the Euler-Lagrange equation $\nabla I(u) = 0$, we often use *gradient descent*, which corresponds to solving the PDE

$$\begin{cases} u_t(x,t) = -\nabla I(u(x,t)), & \text{if } (x,t) \in U \times (0,\infty) \\ u(x,0) = u_0(x), & \text{if } (x,t) \in U \times \{t = 0\}. \end{cases} \tag{2.14}$$

Gradient descent evolves u in the direction that decreases I most rapidly, starting at some initial guess u_0. If we reach a stationary point where $u_t = 0$ then we have found a solution of the Euler-Lagrange equation $\nabla I(u) = 0$. If solutions of the Euler-Lagrange equation are not unique, we may find different solutions depending on the choice of u_0.

To see that gradient descent actually decreases I, let $u(x,t)$ solve (2.14) and compute

$$\begin{aligned}
\frac{d}{dt} I(u) &= \int_U \frac{d}{dt} L(x, u(x,t), \nabla u(x,t))\, dx \\
&= \int_U L_z(x, u, \nabla u) u_t + \nabla_p L(x, u, \nabla u) \nabla u_t\, dx \\
&= \int_U \left(L_z(x, u, \nabla u) - \operatorname{div}\left(\nabla_p L(x, u, \nabla u)\right)\right) u_t\, dx \\
&= \langle \nabla I(u), u_t\rangle_{L^2(U)} \\
&= \langle \nabla I(u), -\nabla I(u)\rangle_{L^2(U)} \\
&= -\|\nabla I(u)\|_{L^2(U)} \\
&\leq 0.
\end{aligned}$$

We used integration by parts in the third line, mimicking the proof of Theorem 1.

Notice that by writing out the Euler-Lagrange equation we can write the gradient descent PDE (2.14) as

$$\begin{cases} u_t + L_z(x,u,\nabla u) - \operatorname{div}\left(\nabla_p L(x,u,\nabla u)\right) = 0, & \text{in } U \times (0,\infty) \\ u = u_0, & \text{on } U \times \{t=0\}. \end{cases} \tag{2.15}$$

Example 8. Gradient descent on the Dirichlet energy (2.6) is the heat equation

$$\begin{cases} u_t - \Delta u = f, & \text{in } U \times (0,\infty) \\ u = u_0, & \text{on } U \times \{t=0\}. \end{cases} \tag{2.16}$$

Thus, solving the heat equation is the fastest way to decrease the Dirichlet energy.

3 Examples continued

We now continue the parade of examples by computing and solving the Euler-Lagrange equations for the examples from Section 1.1.

3.1 Shortest path

Recall for the shortest path problem we wish to minimize

$$I(u) = \int_0^a g(x,u(x))\sqrt{1+u'(x)^2}\,dx,$$

subject to $u(0)=0$ and $u(a)=b$. Here $n=1$ and

$$L(x,z,p) = g(x,z)\sqrt{1+p^2}.$$

Therefore $L_z(x,z,p) = g_z(x,z)\sqrt{1+p^2}$ and $L_p(x,z,p) = g(x,z)(1+p^2)^{-\frac{1}{2}}p$. The Euler-Lagrange equation is

$$g_z(x,u(x))\sqrt{1+u'(x)^2} - \frac{d}{dx}\left(g(x,u(x))(1+u'(x)^2)^{-\frac{1}{2}}u'(x)\right) = 0.$$

This is in general difficult to solve. In the special case that $g(x,z)=1$, $g_z=0$ and this reduces to

$$\frac{d}{dx}\left(\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right) = 0.$$

Computing the derivative yields

$$\frac{\sqrt{1+u'(x)^2}u''(x) - u'(x)(1+u'(x)^2)^{-\frac{1}{2}}u'(x)u''(x)}{1+u'(x)^2} = 0.$$

Multiplying both sides by $\sqrt{1+u'(x)^2}$ we obtain

$$(1+u'(x)^2)u''(x) - u'(x)^2u''(x) = 0.$$

This reduces to $u''(x)=0$, hence the solution is a straight line! This verifies our intuition that the shortest path between two points is a straight line.

3.2 The brachistochrone problem

Recall for the brachistochrone problem we wish to minimize

$$I(u) = \frac{1}{\sqrt{2g}} \int_0^a \sqrt{\frac{1 + u'(x)^2}{-u(x)}}\, dx,$$

subject to $u(0) = 0$ and $u(a) = b$. Here, $n = 1$ and

$$L(x, z, p) = \sqrt{\frac{1 + p^2}{-z}}.$$

Therefore

$$L_p(x, z, p) = \frac{p}{\sqrt{-z(1 + p^2)}}.$$

Notice in this case that L has no x-dependence. Hence we can use the alternative form of the Euler-Lagrange equation (2.7), which yields

$$\sqrt{\frac{1 + u'(x)^2}{-u(x)}} - \frac{u'(x)^2}{\sqrt{-u(x)(1 + u'(x)^2)}} = C$$

for a constant C. Making some algebraic simplifications leads to

$$u(x) + u(x)u'(x)^2 = C, \tag{3.1}$$

where the constant C is different than the one on the previous line. The constant C should be chosen to ensure the boundary conditions hold.

Before solving (3.1), let us note that we can say quite a bit about the solution u from the ODE it solves. First, since $u(a) = b < 0$, the left hand side must be negative somewhere, hence $C < 0$. Solving for $u'(x)^2$ we have

$$u'(x)^2 = \frac{C - u(x)}{u(x)}.$$

This tells us two things. First, since the left hand side is positive, so is the right hand side. Hence $C - u(x) \leq 0$ and so $u(x) \geq C$. If u attains its maximum at an interior point $0 < x < a$ then $u'(x) = 0$ and $u(x) = C$, hence u is constant. This is impossible, so the maximum of u is attained at $x = 0$ and we have

$$C \leq u(x) < 0 \quad \text{for } 0 < x \leq a.$$

This means in particular that we must select $C \leq b$.

Second, as $x \to 0^+$ we have $u(x) \to 0^-$ and hence

$$\lim_{x \to 0^+} u'(x)^2 = \lim_{x \to 0^+} \frac{C - u(x)}{u(x)} = \lim_{t \to 0^-} \frac{C - t}{t} = \infty.$$

Since $x = 0$ is a local maximum of u, we have $u'(x) < 0$ for $x > 0$ small. Therefore

$$\lim_{x \to 0^+} u'(x) = -\infty.$$

This says that the optimal curve starts out vertical.

Third, we can differentiate (3.1) to find that

$$u'(x) + u'(x)^3 + 2u(x)u'(x)u''(x) = 0.$$

Therefore

$$1 + u'(x)^2 = -2u(x)u''(x). \tag{3.2}$$

Since the left hand side is positive, so is the right hand side, and we deduce $u''(x) > 0$. Therefore $u'(x)$ is strictly increasing, and u is a convex function of x. In fact, we can say slightly more. Solving for $u''(x)$ and using (3.1) we have

$$u''(x) = -\frac{1 + u'(x)^2}{2u(x)} = -\frac{(1 + u'(x)^2)^2}{2C} \geq -\frac{1}{2C} > 0.$$

This means that u is uniformly convex, and for large enough x we will have $u'(x) > 0$, so the optimal curve could eventually be *increasing*.

In fact, since u' is strictly increasing, there exists (by the intermediate value theorem) a unique point $x^* > 0$ such that $u'(x^*) = 0$. We claim that u is symmetric about this point, that is

$$u(x^* + x) = u(x^* - x).$$

To see this, write $w(x) = u(x^* + x)$ and $v(x) = u(x^* - x)$. Then

$$w'(x) = u'(x^* + x) \quad \text{and} \quad v'(x) = -u'(x^* - x).$$

Using (3.1) we have

$$w(x) + w(x)w'(x) = u(x^* + x) + u(x^* + x)u'(x^* + x)^2 = C$$

and

$$v(x) + v(x)v'(x) = u(x^* - x) + u(x^* - x)u'(x^* - x)^2 = C.$$

Since $v(0) = u(x^*) = w(0)$, we can use uniqueness of solutions of ODEs to show that $w(x) = v(x)$, which establishes the claim. The point of this discussion is that without explicitly computing the solution, we can say quite a bit both quantitatively and qualitatively about the solutions.

We now solve (3.1). Note we can write

$$u(x) = \frac{C}{1 + u'(x)^2}.$$

Since u' is strictly increasing, the angle θ between the tangent vector $(1, u'(x))$ and the vertical $(0, -1)$ is strictly increasing. Therefore, we can parametrize the curve in terms of this angle θ. Let us write $C(\theta) = (x(\theta), y(\theta))$. Then we have $y(\theta) = u(x)$ and

$$\sin^2\theta = \frac{1}{1 + u'(x)^2}.$$

Therefore
$$y(\theta) = C\sin^2\theta = -\frac{C}{2}(\cos(2\theta) - 1).$$
Since $y(\theta) = u(x)$
$$\frac{dy}{dx} = u'(x) = -\frac{\cos\theta}{\sin\theta}.$$
By the chain rule
$$x'(\theta) = \frac{dx}{d\theta} = \frac{dx}{dy}\frac{dy}{d\theta} = \left(-\frac{\sin\theta}{\cos\theta}\right)(2C\sin\theta\cos\theta) = -2C\sin^2\theta.$$
Therefore
$$x(\theta) = C\int_0^\theta \cos(2\theta) - 1\,dt = -\frac{C}{2}\left(2\theta - \sin(2\theta)\right).$$
This gives an explicit solution for the brachistochrone problem, where θ is just the parameter of the curve.

There is a nice geometrical interpretation of the brachistochrone curve. Notice that
$$\begin{bmatrix} x(\theta) \\ y(\theta) \end{bmatrix} = -\frac{C}{2}\begin{bmatrix} 2\theta \\ -1 \end{bmatrix} - \frac{C}{2}\begin{bmatrix} -\sin(2\theta) \\ \cos(2\theta) \end{bmatrix}.$$
The first term parametrizes the line $y = C/2$, while the second term traverses the circle of radius $r = -C/2$ in the counterclockwise direction. Thus, the curve is traced by a point on the rim of a circular wheel as the wheel rolls along the x-axis. Such a curve is called a *cycloid*.

Notice that the minimum occurs when $\theta = \frac{\pi}{2}$, and $y = C$ and $x = -\frac{C\pi}{2}$. Hence the minima of all brachistochrone curves lie on the line
$$x + \frac{\pi}{2}y = 0.$$
It follow that if $a + \frac{\pi}{2}b > 0$, then the optimal path starts traveling steeply downhill, reaches a low point, and then climbs uphill before arriving at the final point (a, b). If $a + \frac{\pi}{2}b \leq 0$ then the bead is always moving downhill. See Figure 3 for an illustration of the family of brachistochrone curves.

Now, suppose instead of releasing the bead from the top of the curve, we release the bead from some position $(x_0, u(x_0))$ further down (but before the minimum) on the brachistochrone curve. How long does it take the bead to reach the lowest point on the curve? It turns out this time is the same regardless of where you place the bead on the curve! To see why, we recall that conservation of energy gives us
$$\frac{1}{2}mv(x)^2 + mgu(x) = mgu(x_0),$$
where $v(x)$ is the velocity of the bead. Therefore
$$v(x) = \sqrt{2g(u(x_0) - u(x))},$$
and the time to descend to the lowest point is
$$T = \frac{1}{\sqrt{2g}}\int_{x_0}^{-\frac{C\pi}{2}} \sqrt{\frac{1 + u'(x)^2}{u(x_0) - u(x)}}\,dx.$$

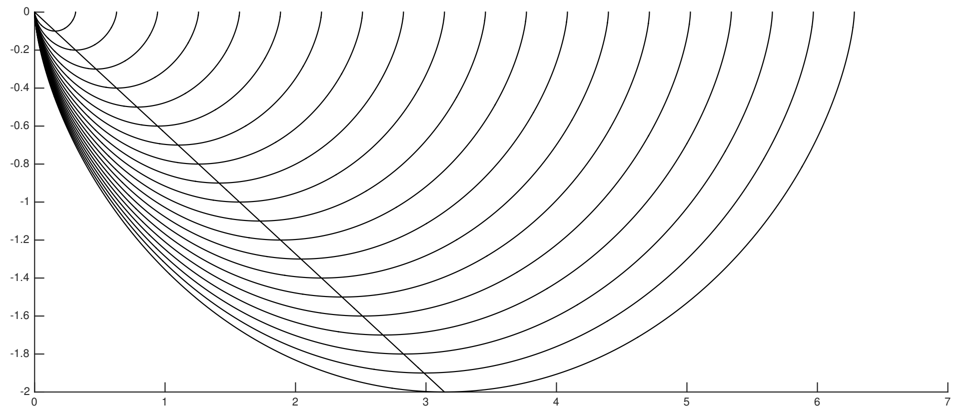


Figure 3: Family of brachistochrone curves. The straight line is the line $x + \frac{\pi}{2}y = 0$ passing through the minima of all brachistochrone curves.

Recall that

$$1 + u'(x)^2 = \frac{1}{\sin^2\theta}, \;\; u(x) = y(\theta) = C\sin^2\theta, \quad \text{and } dx = -2C\sin^2\theta d\theta,$$

where $u(x_0) = y(\theta_0) = C\sin^2\theta_0$. Making the change of variables $x \to \theta$ yields

$$T = \sqrt{\frac{-2C}{g}} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{\sin^2\theta - \sin^2\theta_0}}\, d\theta = \sqrt{\frac{-2C}{g}} \int_{\theta_0}^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{\cos^2\theta_0 - \cos^2\theta}}\, d\theta.$$

Make the change of variables $t = -\cos\theta/\cos\theta_0$. Then $\cos\theta_0 dt = \sin\theta d\theta$ and

$$T = \sqrt{\frac{-2C}{g}} \int_{-1}^{0} \frac{1}{\sqrt{1-t^2}}\, dt.$$

We can integrate this directly to obtain

$$T = \sqrt{\frac{-2C}{g}}\,(\arcsin(0) - \arcsin(-1)) = \pi\sqrt{\frac{-C}{2g}}.$$

Notice this is independent of the initial position x_0 at which the bead is released! A curve with the property that the time taken by an object sliding down the curve to its lowest point is independent of the starting position is called a *tautochrone*, or *isochrone* curve. So it turns out that the tautochrone curve is the same as the brachistochrone curve. The words tautochrone and isochrone are ancient Greek for same-time and equal-time, respectively.

3.3 Minimal surfaces

Recall for the minimal surface problem we wish to minimize

$$I(u) = \int_U \sqrt{1 + |\nabla u|^2}\, dx$$

subject to $u = g$ on ∂U. Here $n \geq 2$ and

$$L(x,z,p) = \sqrt{1+|p|^2} = \sqrt{1+p_1^1+p_2^2+\cdots+p_n^2}.$$

Even though minimal surfaces are defined in dimension $n = 2$, it can still be mathematically interesting to consider the general case of arbitrary dimension $n \geq 2$.

From the form of L we see that $L_z(x,z,p) = 0$ and

$$\nabla_p L(x,z,p) = \frac{p}{\sqrt{1+p^2}}.$$

Therefore, the Euler-Lagrange equation for the minimal surface problem is

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{in } U \tag{3.3}$$

subject to $u = g$ on ∂U. This is called the *minimal surface equation*. Using the chain rule, we can rewrite the PDE as

$$\nabla\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\right)\cdot\nabla u + \frac{1}{\sqrt{1+|\nabla u|^2}}\operatorname{div}(\nabla u) = 0.$$

Notice that

$$\frac{\partial}{\partial x_j}\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\right) = -\frac{1}{2}(1+|\nabla u|^2)^{-\frac{3}{2}}\sum_{i=1}^n 2u_{x_i}u_{x_ix_j}.$$

Therefore the PDE in expanded form is

$$-\frac{1}{(1+|\nabla u|^2)^{\frac{3}{2}}}\sum_{i,j=1}^n u_{x_ix_j}u_{x_i}u_{x_j} + \frac{\Delta u}{\sqrt{1+|\nabla u|^2}} = 0.$$

Multiplying both sides by $(1+|\nabla u|^2)^{\frac{3}{2}}$ we have

$$-\sum_{i,j=1}^n u_{x_ix_j}u_{x_i}u_{x_j} + (1+|\nabla u|^2)\Delta u = 0. \tag{3.4}$$

which is also equivalent to

$$-\nabla u\cdot\nabla^2 u\nabla u + (1+|\nabla u|^2)\Delta u = 0. \tag{3.5}$$

Exercise 2. Show that the plane

$$u(x) = a\cdot x + b$$

solves the minimal surface equation on $U = \mathbb{R}^n$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Exercise 3. Show that for $n = 2$ the Scherk surface

$$u(x) = \log\left(\frac{\cos(x_1)}{\cos(x_2)}\right)$$

solves the minimal surface equation on the box $U = (-\frac{\pi}{2},\frac{\pi}{2})^2$.

Notice that if we specialize to the case of $n = 2$ then we have

$$-u_{x_1x_1}u_{x_1}^2 - 2u_{x_1x_2}u_{x_1}u_{x_2} - u_{x_2x_2}u_{x_2}^2 + (1 + u_{x_1}^2 + u_{x_2}^2)(u_{x_1x_1} + u_{x_2x_2}) = 0,$$

which reduces to

$$(1 + u_{x_2}^2)u_{x_1x_1} - 2u_{x_1x_2}u_{x_1}u_{x_2} + (1 + u_{x_1}^2)u_{x_2x_2} = 0. \tag{3.6}$$

It is generally difficult to find solutions of the minimal surface equation (3.4). It is possible to prove that a solution always exists and is unique, but this is outside the scope of this course.

3.4 Minimal surface of revolution

Suppose we place rings of equal radius $r > 0$ at locations $x = -L$ and $x = L$ on the x-axis. What is the resulting minimal surface formed between the rings? In other words, if we dip the rings into a soapy water solution, what is the shape of the resulting soap bubble?

Here, we may assume the surface is a surface of revolution. Namely, the surface is formed by taking a function $u : [-L, L] \to \mathbb{R}$ with $u(-L) = r = u(L)$ and rotating it around the x-axis. The surface area of a surface of revolution is

$$I(u) = 2\pi \int_{-L}^{L} u(x)\sqrt{1 + u'(x)^2}\, dx.$$

Since $L(x, z, p) = z\sqrt{1 + p^2}$ does not have an x-dependence, the Euler-Lagrange equation can be computed via (2.7) and we obtain

$$u(x)\sqrt{1 + u'(x)^2} - \frac{u'(x)^2u(x)}{\sqrt{1 + u'(x)^2}} = \frac{1}{c}$$

for a constant $c \neq 0$. Multiplying both sides by $\sqrt{1 + u'(x)^2}$ and simplifying we have

$$cu(x) = \sqrt{1 + u'(x)^2}. \tag{3.7}$$

Before solving this, we make the important observation that at a minimum of u, $u'(x) = 0$ and hence $u(x) = \frac{1}{c}$ at the minimum. Since we are using a surface of revolution, we require $u(x) > 0$, hence we must take $c > 0$.

We now square both sides of (3.7) and rearrange to get

$$c^2u(x)^2 - u'(x)^2 = 1. \tag{3.8}$$

Since $u'(x)^2 \geq 0$, we deduce that $c^2u(x)^2 \geq 1$ for all. Since $u(L) = r$ we require

$$c \geq \frac{1}{r}. \tag{3.9}$$

To solve (3.8), we use a clever trick: We differentiate both sides to find

$$2c^2u(x)u'(x) - 2u'(x)u''(x) = 0$$

or

$$u''(x) = c^2u(x).$$

Notice we have converted a nonlinear ODE into a linear ODE! Since $c^2 > 0$ the general solution is

$$u(x) = \frac{A}{2}e^{cx} + \frac{B}{2}e^{-cx}.$$

It is possible to show, as we did for the brachistochrone problem, that u must be an even function. It follows that $A = B$ and

$$u(x) = A\frac{e^{cx} + e^{-cx}}{2} = A\cosh(cx).$$

The minimum of u occurs at $x = 0$, so

$$A = u(0) = \frac{1}{c}.$$

Therefore the solution is

$$u(x) = \frac{1}{c}\cosh(cx). \tag{3.10}$$

This curve is called a *catenoid*, and it turns out to be the same shape as a rope hanging from two posts of equal height under the influence of gravity. An inverted catenoid, or *catenary arch*, has been used in architectural designs since ancient times.

We now need to see if it is possible to select $c > 0$ so that

$$u(-L) = r = u(L).$$

Since u is even, we only need to check that $u(L) = r$. This is equivalent choosing $c > 0$ so that $\cosh(cL) = cr$. Let us set $C = cL$ and $\theta = \frac{r}{L}$. Then we need to choose $C > 0$ such that

$$\cosh(C) = \theta C. \tag{3.11}$$

This equation is not always solvable, and depends on the value of $\theta = \frac{r}{L}$, that is, on the ratio of the radius r of the rings to L, which is half of the separation distance. There is a threshold value θ_0 such that for $\theta > \theta_0$ there are two solutions $C_1 < C_2$ of (3.11). When $\theta = \theta_0$ there is one solution C, and when $\theta < \theta_0$ there are no solutions. See Figure 4 for an illustration. To rephrase this, if $\theta < \theta_0$ or $r < L\theta_0$, then the rings are too far apart and there is no minimal surface spanning the two rings. If $r \geq L\theta_0$ then the rings are close enough together and a minimal surface exists. From numerical computations, $\theta_0 \approx 1.509$.

Now, when there are two solutions $C_1 < C_2$, which one gives the smallest surface area? We claim it is C_1. To avoid complicated details, we give here a heuristic argument to justify this claim. Let $c_1 < c_2$ such that $C_1 = c_1 L$ and $C_2 = c_2 L$. So we have two potential solutions

$$u_1(x) = \frac{1}{c_1}\cosh(c_1 x) \quad \text{and} \quad u_2(x) = \frac{1}{c_2}\cosh(c_2 x).$$

Since $u_1(0) = \frac{1}{c_1} \geq \frac{1}{c_2} = u_2(0)$, we have $u_1 \geq u_2$. In other words, as we increase c the solution decreases. Now, as we pull the rings apart we expect the solution to decrease (the soap bubble becomes thinner), so the value of c should increase as the rings are pulled apart. As the rings are pulled apart L is increasing, so $\theta = r/L$ is decreasing. From Figure 4 we see that C_2 is decreasing as θ decreases. Since $C_2 = c_2 L$ and L is increasing, c_2 must be decreasing as the

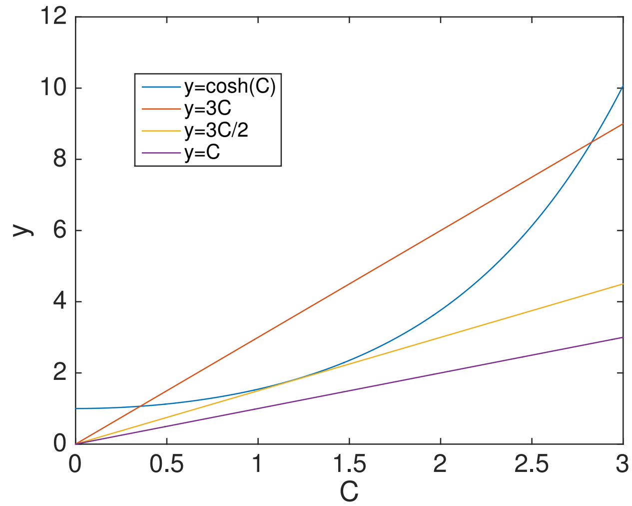


Figure 4: When $\theta > \theta_0$ there are two solutions of (3.11). When $\theta = \theta_0$ there is one solution, and when $\theta < \theta_0$ there are no solutions. By numerical computations, $\theta_0 \approx 1.509$.

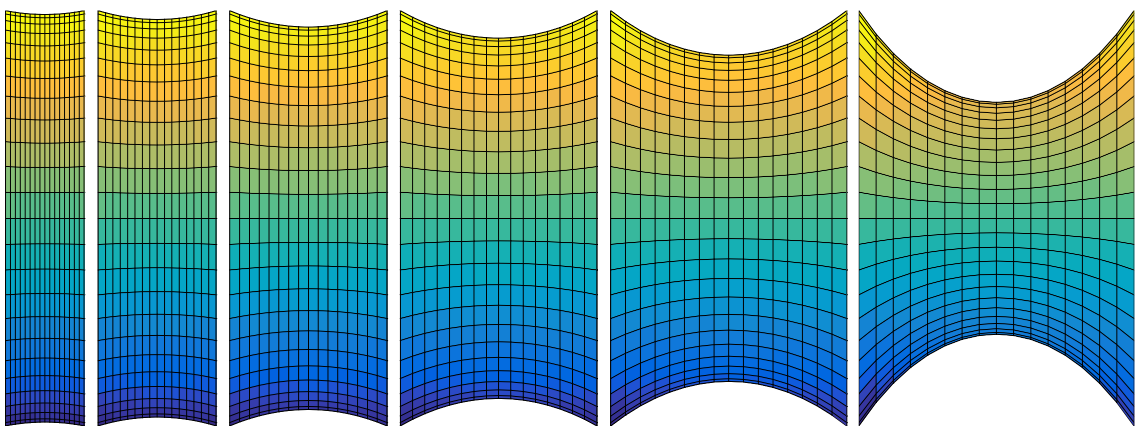


Figure 5: Simulation of the minimal surface of revolution for two rings being slowly pulled apart. The rings are located at $x = -L$ and $x = L$ where L ranges from (left) $L = 0.095$ to (right) $L = 0.662$, and both rings have radius $r = 1$. For larger L the soap bubble will collapse.

rings are pulled apart. In other words, u_2 is *increasing* as the rings are pulled apart, so u_2 is a non-physical solution. The minimal surface is therefore given by u_1. Figure 5 shows the solutions u_1 as the two rings pull apart, and Figure 6 shows non-physical solutions u_2.

We can also explicitly compute the minimal surface area for $c = c_1$. We have

$$I(u) = 2\pi \int_{-L}^{L} u(x)\sqrt{1 + u'(x)^2}\, dx = 4\pi c \int_0^L u(x)^2\, dx, \tag{3.12}$$

where we used the Euler-Lagrange equation (3.7) and the fact that u is even in the last step above. Substituting $u''(x) = c^2u(x)$ and integrating by parts we have

$$\begin{aligned} I(u) &= \frac{4\pi}{c}\int_0^L u''(x)u(x)\,dx \\ &= \frac{4\pi}{c}u'(x)u(x)\Big|_0^L - \frac{4\pi}{c}\int_0^L u'(x)^2\,dx \\ &= \frac{4\pi}{c}u'(L)u(L) - \frac{4\pi}{c}\int_0^L u'(x)^2\,dx. \end{aligned}$$

Using (3.8) we have $u'(x)^2 = c^2u(x)^2 - 1$ and so

$$I(u) = \frac{4\pi u(L)}{c}\sqrt{c^2u(L)^2-1} - \frac{4\pi}{c}\int_0^L c^2u(x)^2 - 1\,dx.$$

Since $u(L) = r$ we have

$$I(u) = \frac{4\pi r}{c}\sqrt{c^2r^2-1} - 4\pi c\int_0^L u(x)^2\,dx + \frac{4\pi}{c}\int_0^L dx.$$

Recalling (3.12) we have

$$I(u) = \frac{4\pi r}{c}\sqrt{c^2r^2-1} - I(u) + \frac{4\pi L}{c}.$$

Solving for $I(u)$ we have

$$I(u) = \frac{2\pi}{c}\left(r\sqrt{c^2r^2-1} + L\right). \tag{3.13}$$

Notice that we have at no point used the explicit formula $u(x) = \frac{1}{c}\cosh(cx)$. We have simply used the ODE that u satisfies, some clever integration by parts, and the boundary condition $u(L) = r$. There is an alternative expression for the surface area. Recall c is chosen so that $cr = \cosh(cL)$. Thus

$$c^2r^2 - 1 = \cosh^2(cL) - 1 = \sinh^2(cL),$$

and we have

$$I(u) = \frac{2\pi}{c}\left(r\sinh(cL) + L\right).$$

While it is not possible to analytically solve for c_1 and c_2, we can numerically compute the values to arbitrarily high precision with our favorite root-finding algorithm. Most root-finding algorithms require one to provide an initial interval in which the solution is to be found. We already showed (see (3.9)) that $c \geq 1/r$. For an upper bound, recall we have the Taylor series

$$\cosh(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2n}}{(2k)!},$$

and so $\cosh(x) \geq \frac{x^2}{2}$. Recall also that c_1 and c_2 are solutions of

$$\cosh(cL) = cr.$$

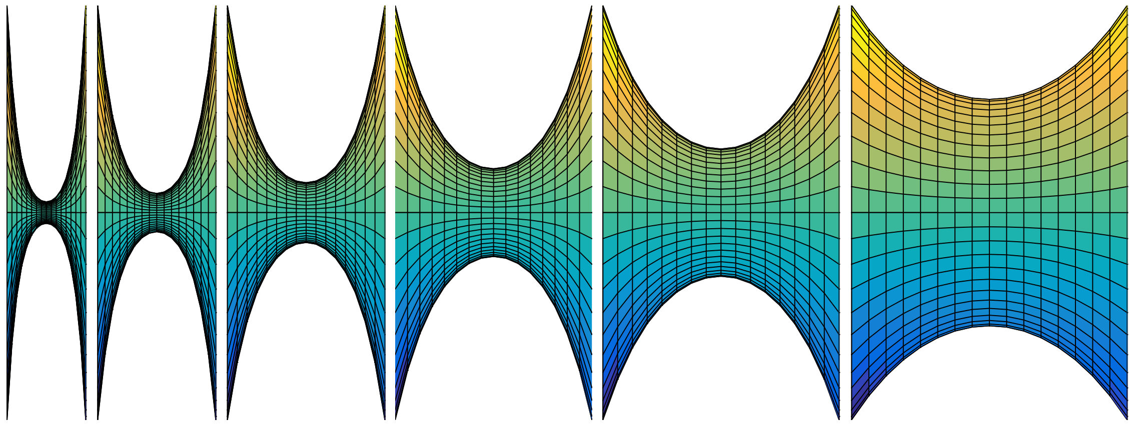


Figure 6: Illustration of solutions of the minimal surface equation that do not minimize surface area. The details are identical to Figure 5, except that we select c_2 instead of c_1. Notice the soap bubble is growing as the rings are pulled apart, which is the opposite of what we expect to occur.

Therefore, if $\frac{c^2L^2}{2} > cr$ we know that $\cosh(cL) > cr$. This gives the bounds

$$\frac{1}{r} \leq c_i \leq \frac{2r}{L^2} \qquad (i = 1, 2).$$

Furthermore, the point c^* where the slope of $c \mapsto \cosh(cL)$ equals r lies between c_1 and c_2. Therefore

$$c_1 < c^* < c_2,$$

where $L\sinh(c^*L) = r$, or

$$c^* = \frac{1}{L}\sinh^{-1}\left(\frac{r}{L}\right).$$

Thus, if $\cosh(c^*L) = c^*r$, then there is exactly one solution $c_1 = c_2 = c^*$. If $\cosh(c^*L) < c^*r$ then there are two solutions

$$\frac{1}{r} \leq c_1 < c^* < c_2 \leq \frac{2r}{L^2}. \tag{3.14}$$

Otherwise, if $\cosh(c^*L) > c^*r$ then there are no solutions. Now that we have the bounds (3.14) we can use any root-finding algorithm to determine the values of c_1 and c_2. In the code I showed in class I used a simple bisection search.

3.5 Image restoration

Recall the total variation (TV) image restoration problem is based on minimizing

$$I(u) = \int_U \frac{1}{2}(f-u)^2 + \lambda|\nabla u|\,dx \tag{3.15}$$

over all $u : U \to \mathbb{R}$, where $U = (0,1)^2$. The function f is the original noisy image, and the minimizer u is the denoised image. Here, the Lagrangian

$$L(x, z, p) = \frac{1}{2}(f(x) - z)^2 + \lambda|p|$$

is not differentiable at $p = 0$. This causes some minor issues with numerical simulations, so it is common to take an approximation of the TV functional that is differentiable. One popular choice is

$$I_\varepsilon(u) = \int_U \frac{1}{2}(f-u)^2 + \lambda\sqrt{|\nabla u|^2 + \varepsilon^2}\,dx, \tag{3.16}$$

where $\varepsilon > 0$ is a small parameter. When $\varepsilon = 0$ we get the TV functional (3.15). In this case the Lagrangian is

$$L_\varepsilon(x, z, p) = \frac{1}{2}(f(x) - z)^2 + \lambda\sqrt{|p|^2 + \varepsilon^2},$$

which is differentiable in both z and p. It is possible to prove that minimizers of I_ε converge to minimizers of I as $\varepsilon \to 0$, but the proof is very technical and outside the scope of this course.

So the idea is to fix some small value of $\varepsilon > 0$ and minimize I_ε. To compute the Euler-Lagrange equation note that

$$L_{\varepsilon,z}(x, z, p) = z - f(x) \quad \text{and} \quad \nabla_p L_\varepsilon(x, z, p) = \frac{\lambda p}{\sqrt{|p|^2 + \varepsilon^2}}.$$

Therefore the Euler-Lagrange equation is

$$u - \lambda\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \varepsilon^2}}\right) = f \quad \text{in } U \tag{3.17}$$

with homogeneous Neumann boundary conditions $\frac{\partial u}{\partial \nu} = 0$ on ∂U. It is almost always impossible to find a solution of (3.17) analytically, so we are left to use numerical approximations.

A standard numerical method for computing solutions of (3.17) is gradient descent, as described in Section 2.1. This is not the fastest or most efficient algorithm, but it is simple to implement and gives nice results. The gradient descent partial differential equation is

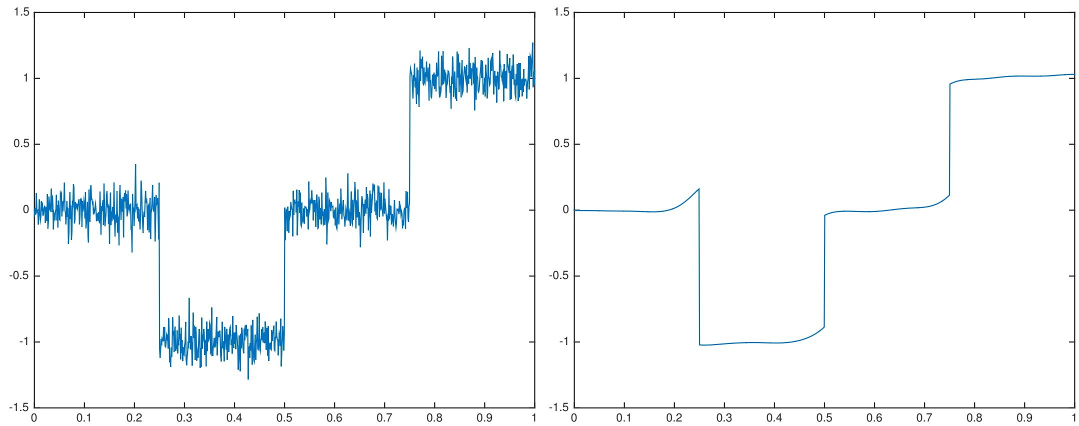
$$u_t + u - \lambda\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{|\nabla u|^2 + \varepsilon^2}}\right) = f \quad \text{for } x \in U,\ t > 0,$$

with initial condition $u(x, 0) = f(x)$ and boundary conditions $\frac{\partial u}{\partial \nu} = 0$ on ∂U. This is a nonlinear heat equation where the thermal conductivity

$$\kappa = \frac{1}{\sqrt{|\nabla u|^2 + \varepsilon^2}}$$

depends on ∇u. We solve the gradient descent equation marching forward in time until $u_t = 0$, which guarantees we have found a solution of the Euler-Lagrange equation (3.17).

Figure 7 shows a one dimensional example of denoising a signal with the TV restoration algorithm. Notice the algorithm can remove noise while preserving the sharp discontinuities in the signal. This suggests that minimizers of the TV restoration functional can have discontinuities. Figure 8 shows an example of TV image restoration of a noisy image of Vincent hall. The top image is the noisy image, the middle image is TV restoration with a small value of λ, and the bottom image is TV restoration with a large value of λ. Notice that as λ is increased, the images becomes smoother, and many fine details are removed. We also observe that for



(a) Noisy signal (b) TV denoising

Figure 7: Example of denoising a noisy signal with the total variations restoration algorithm. Notice the noise is removed while the edges are preserved.

small λ the algorithm is capable of preserving edges and fine details while removing unwanted noise.

The simulations presented in Figures 7 and 8 suggest that minimizers of the TV restoration functional I can be discontinuous. This may present as counter-intuitive, since the derivative of u is very large near a discontinuity and we are in some sense minimizing the derivative. It is important to keep in mind, however, that we are minimizing the *integral* of the derivative, and while the derivative may be large at some points, its integral can still be small.

As an example, let us consider the one dimensional case and ignore the fidelity term, since it does not involve the derivative of u. Hence, we consider the functional

$$J_p(u) = \int_{-1}^{1} |u'(x)|^p \, dx,$$

where $p \geq 1$. The TV functional corresponds to $p = 1$, but it is interesting to consider other values of p to understand why $p = 1$ is preferred in signal processing communities. Suppose we want to minimize J_p subject to $u(-1) = 0$ and $u(1) = 1$. It is not difficult to convince yourself that the minimizer should be an increasing function, so we may write

$$J_p(u) = \int_{-1}^{1} u'(x)^p \, dx,$$

provided we restrict $u'(x) \geq 0$. If $p > 1$ then the Euler-Lagrange equation is

$$\frac{d}{dx}\left(pu'(x)^{p-1}\right) = 0,$$

which expands to

$$p(p-1)u'(x)^{p-2}u''(x) = 0.$$

The straight line $u(x) = \frac{1}{2}x + \frac{1}{2}$ is a solution of the Euler-Lagrange equation, and hence a minimizer since J_p is convex. When $p = 1$ the Euler-Lagrange equation is

$$\frac{d}{dx}(1) = 0$$

Figure 8: Example of TV image restoration: (top) noisy image (middle) TV restoration with small value for λ, and (bottom) TV restoration with large value for λ.

which does not even involve u! This means *every* increasing function is a solution of the Euler-Lagrange equation. A Lagrangian $L(x, z, p)$ for which every function solves the Euler-Lagrange equation is called a *null Lagrangian*, and they have many important applications in analysis.

Notice that when $p = 1$ and u is increasing

$$J_1(u) = \int_{-1}^{1} u'(x)\, dx = u(1) - u(-1) = 1 - 0 = 1.$$

Hence, the functional $J_1(u)$ actually only depends on the boundary values $u(1)$ and $u(-1)$, provided u is increasing. This is the reason why the Euler-Lagrange equation is degenerate; every increasing function satisfying the boundary conditions $u(-1) = 0$ and $u(1) = 1$ is a minimizer. Thus, the TV functional does not care *how* the function gets from $u(-1) = 0$ to $u(1) = 1$, provided the function is increasing. So the linear function $u(x) = \frac{1}{2}x + \frac{1}{2}$ is a minimizer, but so is the sequence of functions

$$u_n(x) = \begin{cases} 0, & \text{if } -1 \le x \le 0 \\ nx, & \text{if } 0 \le x \le \frac{1}{n} \\ 1, & \text{if } \frac{1}{n} \le x \le 1. \end{cases}$$

The function u_n has a sharp transition from zero to one with slope n between $x = 0$ and $x = 1/n$. For each n

$$J_1(u_n) = \int_0^{\frac{1}{n}} n\, dx = 1,$$

so each u_n is a minimizer. The pointwise limit of u_n as $n \to \infty$ is the Heaviside function

$$H(x) = \begin{cases} 0, & \text{if } -1 \le x \le 0 \\ 1, & \text{if } 0 \le x \le 1. \end{cases}$$

So in some sense, the discontinuous function $H(x)$ is also a minimizer. Indeed, we can compute

$$J_1(H) = \int_{-1}^{1} H'(x)\, dx = \int_{-1}^{1} \delta(x)\, dx = 1,$$

where $\delta(x)$ is the Delta function. This explains why minimizers of the TV functional can admit discontinuities.

Notice that if $p > 1$ then

$$J_p(u_n) = \int_0^{\frac{1}{n}} n^p \ dx = n^{p-1},$$

hence $J_p(u_n) \to \infty$ as $n \to \infty$. This means that a discontinuous function cannot minimize J_p for $p > 1$, and the only sensible value for $J_p(H)$ is ∞ when $p > 1$. Thus, if we used a version of TV restoration where $|\nabla u|^p$ appeared with $p > 1$, we would not expect that edges and fine details would be preserved, since discontinuities have infinite cost in this case.

3.6 Image segmentation

Recall in the image segmentation problem we aim to minimize

$$I(u,a,b) = \int_U H(u)(f-a)^2 + (1-H(u))\,(f-b)^2 + \lambda\delta(u)|\nabla u|\,dx$$

over $u : U \to \mathbb{R}$ and real numbers a and b. Let us assume for the moment that a and b are fixed, and I is a function of only u.

The Lagrangian

$$L(x,z,p) = H(z)(f(x)-a)^2 + (1-H(z))(f(x)-b)^2 + \lambda\delta(z)|p|$$

is not even continuous, due to the presence of the Heaviside function $H(z)$ and the delta function $\delta(z)$. This causes problems numerically, hence in practice we usually replace L by a smooth approximation. For $\varepsilon > 0$ we define the smooth approximate Heaviside function

$$H_\varepsilon(x) = \frac{1}{2}\left(1 + \frac{2}{\pi}\arctan\left(\frac{x}{\varepsilon}\right)\right). \tag{3.18}$$

The approximation to δ is then

$$\delta_\varepsilon(x) := H_\varepsilon'(x) = \frac{1}{\pi}\frac{\varepsilon}{\varepsilon^2 + x^2}. \tag{3.19}$$

We then form the approximation

$$I_\varepsilon(u) = \int_U H_\varepsilon(u)(f-a)^2 + (1-H_\varepsilon(u))\,(f-b)^2 + \lambda\delta_\varepsilon(u)|\nabla u|\,dx. \tag{3.20}$$

The Lagrangian for I_ε is

$$L_\varepsilon(x,z,p) = H_\varepsilon(z)(f(x)-a)^2 + (1-H_\varepsilon(z))(f(x)-b)^2 + \lambda\delta_\varepsilon(z)|p|.$$

Therefore

$$L_{\varepsilon,z}(x,z,p) = \delta_\varepsilon(z)\left((f(x)-a)^2 - (f(x)-b)^2\right) + \lambda\delta_\varepsilon'(z)|p|$$

and

$$\nabla_p L_\varepsilon(x,z,p) = \frac{\lambda\delta_\varepsilon(z)p}{|p|}.$$

By the chain and product rules

$$\begin{aligned}
\operatorname{div}\left(\nabla_p L(x,u(x),\nabla u(x))\right) &= \lambda\operatorname{div}\left(\frac{\delta_\varepsilon(u(x))\nabla u(x)}{|\nabla u(x)|}\right)\\
&= \lambda\frac{\nabla\delta_\varepsilon(u(x))\cdot\nabla u(x)}{|\nabla u(x)|} + \lambda\operatorname{div}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right)\\
&= \lambda\frac{\delta_\varepsilon'(u(x))\nabla u(x)\cdot\nabla u(x)}{|\nabla u(x)|} + \lambda\delta_\varepsilon(u(x))\operatorname{div}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right)\\
&= \lambda\delta_\varepsilon'(u(x))|\nabla u(x)| + \lambda\delta_\varepsilon(u(x))\operatorname{div}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right)
\end{aligned}$$

Therefore, the Euler-Lagrange equation is

$$\delta_\varepsilon(u)\left[(f-a)^2-(f-b)^2-\lambda\,\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right]=0 \ \text{ in } U \tag{3.21}$$

subject to homogeneous Neumann boundary conditions $\frac{\partial u}{\partial \nu}=0$ on ∂U.

As with the image restoration problem, it is nearly impossible to solve the Euler-Lagrange equation (3.21) analytically. Thus we are left to devise numerical algorithms to find solutions. Here, we are minimizing over u, a, and b, which is a situation we have not encountered before. Note that if u is fixed, then minimizing with respect to a and b is easy. Indeed, differentiating I_ε with respect to a yields

$$0=-2\int_U H_\varepsilon(u)(f-a)\,dx.$$

Therefore the optimal value for a is

$$a=\frac{\int_U H_\varepsilon(u)f\,dx}{\int_U H_\varepsilon(u)\,dx}, \tag{3.22}$$

which is approximately the average of f in the region where $u>0$. Similarly, if u is fixed, the optimal choice of b is

$$b=\frac{\int_U(1-H_\varepsilon(u))f\,dx}{\int_U 1-H_\varepsilon(u)\,dx}. \tag{3.23}$$

Since it is easy to minimize over a and b, the idea now is to consider an alternating minimization algorithm, whereby one fixes $a,b\in\mathbb{R}$ and takes a small gradient descent step in the direction of minimizing I_ε with respect to u, and then one freezes u and updates a and b according to (3.22) and (3.23). We repeat this iteratively until the values of a, b, and u remain unchanged with each new iteration.

Gradient descent on I_ε with respect to u is the partial differential equation

$$u_t+\delta_\varepsilon(u)\left[(f-a)^2-(f-b)^2-\lambda\,\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)\right]=0 \ \text{ for } x\in U,\ t>0 \tag{3.24}$$

subject to homogeneous Neumann boundary conditions $\frac{\partial u}{\partial \nu}=0$ on ∂U and an initial condition $u(x,0)=u_0(x)$. As with the image restoration problem, we normally replace the non-differentiable norm $|\nabla u|$ by a smooth approximation, so instead we solve the partial differential equation

$$u_t+\delta_\varepsilon(u)\left[(f-a)^2-(f-b)^2-\lambda\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{|\nabla u|^2+\varepsilon^2}}\right)\right]=0 \ \text{ for } x\in U,\ t>0. \tag{3.25}$$

At each iteration of solving (3.25) numerically, we update the values of a and b according to (3.22) and (3.23).

Figure 9 shows the result of segmenting the cameraman image. The bottom four images in the figure show the evolution of the zero level set of u throughout the gradient descent procedure resulting in the segmentation obtained in the lower right image. Figure 10 shows that results of segmenting blurry and noisy versions of the cameraman image, to illustrate that the algorithm is robust to image distortions.

Figure 9: Illustration of gradient descent for segmenting the cameraman image. Top left is the original image, and top right is the initialization of the gradient descent algorithm. The lower four images show the evolution of the zero level set of u.

Figure 10: Segmentation of clean, blurry, and noisy versions of the cameraman image.

4 The Lagrange multiplier

There are many problems in the calculus of variations that involve constraints on the feasible minimizers. A classic example is the *isoperimetric problem*, which corresponds to finding the shape of a simple closed curve that maximizes the enclosed area given the curve has a fixed length ℓ. Here we are maximizing the area enclosed by the curve subject to the constraint that the length of the curve is equal to ℓ.

Let I and J be functionals defined by

$$I(u) = \int_U L(x, u(x), \nabla u(x))\, dx \quad \text{and} \quad J(u) = \int_U H(x, u(x), \nabla u(x))\, dx.$$

We consider the problem of minimizing $I(u)$ subject to the constraint $J(u) = 0$. The Lagrange multiplier method gives necessary conditions that must be satisfied by any minimizer.

Theorem 3 (Lagrange multiplier)**.** *Suppose that $u \in C^2(\overline{U})$ satisfies $J(u) = 0$ and*

$$I(u) \le I(v) \tag{4.1}$$

for all $v \in C^2(\overline{U})$ with $v = u$ on ∂U and $J(v) = 0$. Then there exists a real number λ such that

$$\nabla I(u) + \lambda \nabla J(u) = 0 \quad \text{in } U. \tag{4.2}$$

Here, ∇I and ∇J are the functional gradients of I and J, respectively, defined by (2.12).

Remark 4. The number λ in Theorem 3 is called a *Lagrange multiplier.*

Proof. We will give a short sketch of the proof. Let $\varphi \in C_c^\infty(U)$. Then as in Theorem 1

$$\frac{d}{dt}\Big|_{t=0} I(u + t\varphi) = \langle \nabla I(u), \varphi \rangle_{L^2(U)} \quad \text{and} \quad \frac{d}{dt}\Big|_{t=0} J(u + t\varphi) = \langle \nabla J(u), \varphi \rangle_{L^2(U)}.$$

Suppose that $\langle \nabla J(u), \varphi \rangle_{L^2(U)} = 0$. Then, up to a small approximation error

$$0 = J(u) = J(u + t\varphi)$$

for small t. Since $\varphi = 0$ on ∂U, we also have $u = u + t\varphi$ on ∂U. Thus by hypothesis

$$I(u) \le I(u + t\varphi) \quad \text{for all small } t.$$

Therefore, $t \mapsto I(u + t\varphi)$ has a minimum at $t = 0$ and so

$$\langle \nabla I(u), \varphi \rangle_{L^2(U)} = \frac{d}{dt}\Big|_{t=0} I(u + t\varphi) = 0.$$

Hence, we have shown that for all $\varphi \in C_c^\infty(U)$

$$\langle \nabla J(u), \varphi \rangle_{L^2(U)} = 0 \implies \langle \nabla I(u), \varphi \rangle_{L^2(U)} = 0.$$

This says that $\nabla I(u)$ is orthogonal to everything that is orthogonal to $\nabla J(u)$. Intuitively this must imply that $\nabla I(u)$ and $\nabla J(u)$ are co-linear; we give the proof below.

We now have three cases.

1. If $\nabla J(u) = 0$ then $\langle \nabla I(u), \varphi \rangle_{L^2(U)} = 0$ for all $\varphi \in C^\infty(U)$, and by the vanishing lemma $\nabla I(u) = 0$. Here we can choose any real number for λ.
2. If $\nabla I(u) = 0$ then we can take $\lambda = 0$ to complete the proof.
3. Now we can assume $\nabla I(u) \neq 0$ and $\nabla J(u) \neq 0$. Define

$$\lambda = -\frac{\langle \nabla I(u), \nabla J(u) \rangle_{L^2(U)}}{\|\nabla J(u)\|_{L^2(U)}} \quad \text{and} \quad v = \nabla I(u) + \lambda \nabla J(u).$$

By the definition of λ we can check that

$$\langle \nabla J(u), v \rangle_{L^2(U)} = 0.$$

Therefore $\langle \nabla I(u), v \rangle_{L^2(U)} = 0$ and we have

$$\begin{aligned} 0 &= \langle \nabla I(u), v \rangle_{L^2(U)} \\ &= \langle v - \lambda \nabla J(u), v \rangle_{L^2(U)} \\ &= \langle v, v \rangle_{L^2(U)} - \lambda \langle \nabla J(u), v \rangle_{L^2(U)} \\ &= \|v\|^2_{L^2(U)}. \end{aligned}$$

Therefore $v = 0$ and so $\nabla I(u) + \lambda \nabla J(u) = 0$. This completes the proof. □

Remark 5. Notice that (4.2) is equivalent to the necessary conditions minimizers for the augmented functional

$$K(u) = I(u) + \lambda J(u).$$

4.1 Isoperimetric inequality

Let C be a simple closed curve in the plane $\mathbb{R}^2$ of length ℓ, and let A denote the area enclosed by C. How large can A be, and what shape of curve yields the largest enclosed area? This question, which is called the *isoperimetric problem*, and similar questions have intrigued mathematicians for many thousands of years. The origin of the isoperimetric problem can be traced back to a Greek mathematician Zenodorus sometime in the second century B.C.E.

Let us consider a few examples. If C is a rectangle of width w and height h, then $\ell = 2(w+h)$ and $A = wh$. Since $w = \frac{1}{2}\ell - h$ we have

$$A = \frac{1}{2}\ell h - h^2,$$

where $h < \frac{1}{2}\ell$. The largest area for this rectangle is attained when $\frac{1}{2}\ell - 2h = 0$, or $h = \frac{1}{4}\ell$. That is, the rectangle is a square! The area of the square is

$$A = \frac{\ell^2}{16}.$$

Can we do better? We can try regular polygons with more sides, such as a pentagon, hexagon, etc., and we will find that the area increases with the number of sides. In the limit as the number of sides tends to infinity we get a circle, so perhaps the circle is a good guess for the optimal shape. If C is a circle of radius $r > 0$ then $2\pi r = \ell$, so $r = \frac{\ell}{2\pi}$ and

$$A = \pi r^2 = \frac{\ell^2}{4\pi}.$$

This is clearly better than the square, since $\pi < 4$.

The question again is: Can we do better still? Is there some shape we have not thought of that would give larger area than a circle while having the same perimeter? We might expect the answer is no, but lack of imagination is not a convincing proof.

We will prove shortly that, as expected, the circle gives the largest area for a fixed perimeter. Thus for any simple closed curve C we have the *isoperimetric inequality*

$$\boxed{4\pi A \leq \ell^2,} \tag{4.3}$$

where equality holds only when C is a circle of radius $r = \frac{\ell}{2\pi}$.

We give here a short proof of the isoperimetric inequality (4.3) using the Lagrange multiplier method in the calculus of variations. Let the curve C be parametrized by $(x(t), y(t))$ for $t \in [0,1]$. For notational simplicity we write $x = x_1$ and $y = x_2$ in this section. Since C is a simple closed curve, we may also assume without loss of generality that

$$x(0) = y(0) = x(1) = y(1) = 0. \tag{4.4}$$

Let U denote the interior of C. Then the area enclosed by the curve C is

$$A(x,y) = \int_U dx = \int_U \operatorname{div}(F)\, dx,$$

where F is the vector field $F(x,y) = \frac{1}{2}(x,y)$. By the divergence theorem

$$A(x,y) = \int_{\partial U} F \cdot \nu \, dS,$$

where ν is the outward normal to ∂U. Since $\partial U = C$ we have

$$\nu(t) = \frac{(y'(t), -x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}},$$

and

$$dS = \sqrt{x'(t)^2 + y'(t)^2}\, dt,$$

provided we take the curve to have positive orientation. Therefore

$$A(x, y) = \int_0^1 \frac{1}{2}(x(t), y(t)) \cdot (y'(t), -x'(t))\, dt = \frac{1}{2}\int_0^1 x(t)y'(t) - x'(t)y(t)\, dt.$$

The length of C is given by

$$\ell(x, y) = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Thus, we wish to find functions $x, y : [0, 1] \to \mathbb{R}$ that maximize $A(x, y)$ subject to $\ell(x, y) = \ell$ and the boundary conditions (4.4).

The area and length functionals depend on two functions $x(t)$ and $y(t)$, which is a situation we have not encountered yet. Similar to the case of partial differentiation of functions on $\mathbb{R}^n$, we can freeze one input, say $y(t)$, and take the functional gradient with respect to $x(t)$. So we treat A and ℓ as functions of $x(t)$ only, and hence $z = x(t)$ and $p = x'(t)$. The gradient $\nabla_x A$, or Euler-Lagrange equation, is then given by

$$\nabla_x A(x, y) = \frac{1}{2}y'(t) - \frac{d}{dt}\left(-\frac{1}{2}y(t)\right) = y'(t)$$

while $\nabla_x \ell(x, y)$ is given by

$$\nabla_x \ell(x, y) = -\frac{d}{dt}\left(\frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}\right).$$

Similarly

$$\nabla_y A(x, y) = -\frac{1}{2}x'(t) - \frac{d}{dt}\left(\frac{1}{2}x(t)\right) = -x'(t),$$

and

$$\nabla_y \ell(x, y) = -\frac{d}{dt}\left(\frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}\right).$$

Then the gradients of A and ℓ are defined as

$$\nabla A(x, y) = \begin{bmatrix} \nabla_x A(x, y) \\ \nabla_y A(x, y) \end{bmatrix} \quad \text{and} \quad \nabla \ell(x, y) = \begin{bmatrix} \nabla_x \ell(x, y) \\ \nabla_y \ell(x, y) \end{bmatrix}.$$

Following (4.2), the necessary conditions for our constrained optimization problem are

$$\nabla A(x, y) + \lambda \nabla \ell(x, y) = 0,$$

where λ is a Lagrange multiplier. This is a set of two equations

$$y'(t) - \frac{d}{dt}\left(\frac{\lambda x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}\right) = 0,$$

and

$$-x'(t) - \frac{d}{dt}\left(\frac{\lambda y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}\right) = 0.$$

Integrating both sides we get

$$y(t) - \frac{\lambda x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} = a \quad \text{and} \quad x(t) + \frac{\lambda y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}} = b,$$

for constants a and b. Therefore

$$\begin{aligned}(x(t) - a)^2 + (y(t) - b)^2 &= \left(-\frac{\lambda y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}\right)^2 + \left(\frac{\lambda x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}\right)^2 \\ &= \frac{\lambda^2 y'(t)^2}{x'(t)^2 + y'(t)^2} + \frac{\lambda^2 x'(t)^2}{x'(t)^2 + y'(t)^2} \\ &= \lambda^2 \frac{x'(t)^2 + y'(t)^2}{x'(t)^2 + y'(t)^2} \\ &= \lambda^2.\end{aligned}$$

This means that the curve $C(t) = (x(t), y(t))$ is a circle of radius λ centered at (a, b). Hence, as we expected, the circle is shape with largest area given a fixed perimeter. Since the perimeter is $\ell(x, y) = \ell$ we have $\lambda = \frac{\ell}{2\pi}$. Since $x(0) = y(0) = 0$, a and b must be chosen to satisfy

$$a^2 + b^2 = \lambda^2 = \frac{\ell^2}{4\pi^2}.$$

That is, the circle must pass through the origin, due to our boundary conditions (4.4).

5 Sufficient conditions

So far, we have only shown that the Euler-Lagrange equation (2.3) is a necessary condition that a minimizer must satisfy. This means that if u minimizes $I(u)$, then u must satisfy the Euler-Lagrange equation. However, solutions of the Euler-Lagrange equation need not always be minimizers. They could be also maximizers or even saddle points as well. Consider, for example, the minimal surface of revolution problem discussed in Section 3.4, where we found two solutions of the Euler-Lagrange equation, but only one solution yielded the least area. The other solution is clearly not a minimizer. For a simpler example, consider the function $f(x) = x^3$. The point $x = 0$ is a critical point, since $f'(0) = 0$, but $x = 0$ is neither a minimum nor maximum of f—it is a saddle point.

So a natural question concerns sufficient conditions for a solution of the Euler-Lagrange equation to be a minimizer or maximizer. Another question concerns whether $I(u)$ actually attains its minimum or maximum. The following example shows that the minimum or maximum need not be attained.

Example 9. Consider the problem of minimizing

$$I(u) = \int_0^1 u(x)^2 + (u'(x)^2 - 1)^2 \, dx,$$

subject to $u(0) = 0 = u(1)$. We claim there is no solution to this problem. To see this, let $u_k(x)$ be a sawtooth wave of period $2/k$ and amplitude $1/k$. That is, $u_k(x) = x$ for $x \in [0, 1/k]$, $u_k(x) = 2/k - x$ for $x \in [1/k, 2/k]$, $u_k(x) = x - 2/k$ for $x \in [2/k, 3/k]$ and so on. The function u_k satisfies $u_k'(x) = 1$ or $u_k'(x) = -1$ at all but a finite number of points of non-differentiability. Also $0 \leq u_k(x) \leq 1/k$. Therefore

$$I(u_k) \leq \int_0^1 \frac{1}{k^2}\,dx = \frac{1}{k^2}.$$

If the minimum exists, then

$$0 \leq \min_u I(u) \leq \frac{1}{k^2}$$

for all natural numbers k. Therefore $\min_u I(u)$ would have to be zero. However, there is no function u for which $I(u) = 0$. Such a function would have to satisfy $u(x)^2 = 0$ and $u'(x)^2 = 1$ for all x, which are not compatible conditions. Therefore, there is no minimizer of I, and hence no solution to this problem.

The lack of differentiability of the sequence u_k is not a serious problem for the example. Indeed, we could smooth out u_k without changing the fact that there exists u with $I(u)$ arbitrarily close to zero.

To understand what went wrong here, it is useful to consider functions of a single variable $f(x)$. If $f'(x) = 0$ then x is a critical point, and if $f''(x) > 0$ then we know that x is a local minimizer of f. This is often called the *second derivative test*. Recall that f is convex if $f''(x) > 0$. So in the calculus of variations, we might expect convexity to play a role.

Our functionals are of the form

$$I(u) = \int_U L(x, u(x), \nabla u(x))\,dx.$$

The integral is a linear operation, so we should ask that L is convex in u and ∇u. Notice in Example 9 that

$$L(x, z, p) = z^2 + (p^2 - 1)^2$$

is *not* convex in p (sketch the graph of $p \mapsto (p^2 - 1)^2$). This is the source of the difficulty in this problem.

5.1 Basic theory of convex functions

Before proceeding, we review some theory of convex functions.

Definition 2. A function $u : \mathbb{R}^n \to \mathbb{R}$ is *convex* if

$$u(\lambda x + (1 - \lambda)y) \leq \lambda u(x) + (1 - \lambda)u(y) \tag{5.1}$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in (0, 1)$.

Lemma 1. *Let $u : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Then u is convex if and only if $u''(x) \geq 0$ for all $x \in \mathbb{R}$.*

Proof. The proof is split into two parts.

1. Assume u is convex. Let $x_0 \in \mathbb{R}$ and set $\lambda = \frac{1}{2}$, $x = x_0 - h$, and $y = x_0 + h$ for a real number h. Then

$$\lambda x + (1-\lambda)y = \frac{1}{2}(x_0 - h) + \frac{1}{2}(x_0 + h) = x_0,$$

and the convexity condition (5.1) yields

$$u(x_0) \leq \frac{1}{2}u(x_0 - h) + \frac{1}{2}u(x_0 + h).$$

Therefore

$$u(x_0 - h) - 2u(x_0) + u(x_0 + h) \geq 0$$

for all h, and so

$$u''(x_0) = \lim_{h \to 0} \frac{u(x_0 - h) - 2u(x_0) + u(x_0 + h)}{h^2} \geq 0.$$

2. Assume that $u''(x) \geq 0$ for all $x \in \mathbb{R}$. Let $x, y \in \mathbb{R}^n$ and $\lambda \in (0,1)$, and set $x_0 = \lambda x + (1-\lambda)y$. Define

$$L(z) = u(x_0) + u'(x_0)(z - x_0).$$

Since $u''(z) \geq 0$ for all z, the Taylor series inequality (A.4) with $\theta = 0$ yields $u(z) \geq L(z)$ for all z. Therefore

$$u(\lambda x + (1-\lambda)y) = u(x_0) = \lambda L(x) + (1-\lambda)L(y) \leq \lambda u(x) + (1-\lambda)u(y),$$

and so u is convex. $\square$

Theorem 4. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Then u is convex if and only if*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_i x_j}(x) v_i v_j \geq 0 \quad \textit{for all } x, v \in \mathbb{R}^n. \tag{5.2}$$

Proof. By the definition of convexity, Definition 2, u is convex if and only if the restriction of u to every line in $\mathbb{R}^n$ is convex. That is, u is convex if and only if for every $x, v \in \mathbb{R}^n$ the function

$$g(t) := u(x + tv)$$

is convex in the single variable $t \in \mathbb{R}$. By Lemma 1 g is convex if and only if $g''(t) \geq 0$ for all t. As we computed in (A.13)

$$g''(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_i x_j}(x + tv) v_i v_j.$$

Hence, u is convex if and only if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_i x_j}(x + tv) v_i v_j \geq 0$$

for all $x, v \in \mathbb{R}^n$ and $t \in \mathbb{R}$. This completes the proof. $\square$

By Theorem 4 and Eq. (A.14) with $\theta = 0$, if u is convex or if u satisfies (5.2) then

$$u(y) \geq u(x) + \nabla u(x) \cdot (y - x) \tag{5.3}$$

holds for all $x, y \in \mathbb{R}^n$. Thus a convex function lies above all of its tangent planes.

5.2 Convexity is a sufficient condition

We now show that a sufficient condition for a critical point, or solution of the Euler-Lagrange equation (2.3), to be a minimum is that the mapping $(z,p) \to L(x,z,p)$ is convex for each $x \in U$. In light of Theorem 4, this means that

$$L_{zz}(x,z,p)t^2 + 2\sum_{i=1}^n L_{zp_i}(x,z,p)tv_i + \sum_{i=1}^n\sum_{j=1}^n L_{p_ip_j}(x,z,p)v_iv_j \geq 0 \tag{5.4}$$

for all $t, z \in \mathbb{R}$ and $x, v, p \in \mathbb{R}^n$.

Sometimes (5.4) is referred to as *joint convexity* in z and p, to distinguish it from the condition that $z \mapsto L(x,z,p)$ and $p \mapsto L(x,z,p)$ are convex. The two notions are not equivalent.

Example 10. The function $u(x) = x_1x_2$ is not convex, since

$$\sum_{i=1}^2\sum_{j=1}^2 u_{x_ix_j}v_iv_j = 2v_1v_2$$

is clearly not positive for, say, $v_1 = 1$ and $v_2 = -1$. However, $x_1 \mapsto u(x_1,x_2)$ is convex, since $u_{x_1x_1} = 0$, and $x_2 \mapsto u(x_1,x_2)$ is convex, since $u_{x_2x_2} = 0$. Therefore, convexity of $x \mapsto u(x)$ is not equivalent to convexity in each variable x_1 and x_2 independently.

The following theorem shows that solutions of the Euler-Lagrange equation are minimizers whenever $(z,p) \mapsto L(x,z,p)$ is convex.

Theorem 5. *Assume $(z,p) \mapsto L(x,z,p)$ is convex for each $x \in U$. Let $u \in C^1(\overline{U})$ be a weak solution of the Euler-Lagrange equation (2.3), as defined in Remark 3. Then $I(u) \leq I(v)$ for all $v \in C^1(\overline{U})$ with $u = v$ on ∂U.*

Proof. Since $(z,p) \mapsto L(x,z,p)$ is convex, we can use (5.3) to show that

$$L(x,v,q) \geq L(x,z,p) + L_z(x,z,p)(v-z) + \nabla_p L(x,z,p)\cdot(q-p)$$

for all $x,p,q \in \mathbb{R}^n$ and $v,z \in \mathbb{R}$. Let $v \in C^1(\overline{U})$ such that $u = v$ on ∂U, and write $p = \nabla u(x)$, $q = \nabla v(x)$, $z = u(x)$, and $v = v(x)$ in the above to obtain

$$L(x,v,\nabla v) \geq L(x,u,\nabla u) + L_z(x,z,p)(v-u) + \nabla_p L(x,u,\nabla u)\cdot(\nabla v - \nabla u).$$

We now integrate both sides over U and write $\varphi = v - u$ to deduce

$$I(v) \geq I(u) + \int_U L_z(x,z,p)\varphi + \nabla_p L(x,u,\nabla u)\cdot\nabla\varphi\,dx.$$

Since u is a weak solution of the Euler-Lagrange equation (2.3) and $\varphi = 0$ on ∂U we have

$$\int_U L_z(x,z,p)\varphi + \nabla_p L(x,u,\nabla u)\cdot\nabla\varphi\,dx = 0$$

and so $I(v) \geq I(u)$. □

Exercise 4. Show that $L(x,z,p) = \frac{1}{2}|p|^2 - zf(x)$ is jointly convex in z and p.

Exercise 5. Show that $L(x,z,p) = zp_1$ is not jointly convex in z and p.

A Mathematical preliminaries

A.1 Integration

Many students are accustomed to using different notation for integration in different dimensions. For example, integration along the real line in $\mathbb{R}$ is usually written

$$\int_a^b u(x)\,dx,$$

while integration over a region $U \subset \mathbb{R}^2$ is written

$$\iint_U u(x,y)\,dxdy \quad \text{or} \quad \iint_U u(\mathbf{x})\,d\mathbf{x},$$

where $\mathbf{x} = (x,y)$. Integration over a volume $U \subset \mathbb{R}^3$ is then written as

$$\iiint_U u(x,y,z)\,dxdydz \quad \text{or} \quad \iiint_U u(\mathbf{x})\,d\mathbf{x}.$$

This becomes cumbersome when we consider problems in an arbitrary number of dimensions n. First, what shall we call the fourth variable (well, sometimes it is time t, but then what is the fifth?). Second, integration over a set $U \subset \mathbb{R}^n$ is written with n integrals $\int\int\cdots\int_U$. This becomes a notational inconvenience and makes it difficult to communicate mathematically.

In these notes, we use x (or y or z) for a point in $\mathbb{R}^n$, so $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. We write

$$u(x) = u(x_1, x_2, \ldots, x_n)$$

for a function $u : \mathbb{R}^n \to \mathbb{R}$. The integration of u over a domain $U \subset \mathbb{R}^n$ is then written

$$\int_U u(x)\,dx \quad \text{or just} \quad \int_U u\,dx,$$

where $dx = dx_1 dx_2 \cdots dx_n$. This notation has the advantage of being far more compact without losing the meaning.

We may interpret the integral $\int_U u\,dx$ in the Riemann or Lebesgue sense. To interpret the integral in the Riemann sense, we partition the domain into M rectangles and approximate the integral by a Riemann sum

$$\int_U u\,dx \approx \sum_{k=1}^M u(x_k)\Delta x_k,$$

where $x_k \in \mathbb{R}^n$ is a point in the k^{th} rectangle, and $\Delta x_k := \Delta x_{k,1}\Delta x_{k,2}\cdots\Delta x_{k,n}$ is the n-dimensional volume (or measure) of the k^{th} rectangle ($\Delta_{k,i}$ for $i = 1,\ldots,n$ are the side lengths of the k^{th} rectangle). Then the Riemann integral is defined by taking the limit as the largest side length in the partition tends to zero (provided the limit exists and does not depend on the choice of partition or points x_k).

Notice here that $x_k = (x_{k,1},\ldots,x_{k,n}) \in \mathbb{R}^n$ is a point in $\mathbb{R}^n$, and not the k^{th} entry of x. There is a slight abuse of notation here; the reader will have to discern from the context which is implied.

The Lebesgue integral is more robust that the Riemann integral and is preferred in more rigorous texts. We refer the reader to a standard graduate analysis book (such as Rudin's "Real and Complex Analysis") for the definition of the Lebesgue integral.

If $S \subset \mathbb{R}^n$ is an $n-1$ dimensional (or possibly lower dimensional) surface, we write the surface integral of u over S as

$$\int_S u(x)\, dS(x).$$

Here, $dS(x)$ is the surface area element at $x \in S$.

A.2 Inequalities

For $x \in \mathbb{R}^n$ the norm of x is

$$|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

When $n = 2$ or $n = 3$, $|x - y|$ is the usual Euclidean distance between x and y. The dot product between $x, y \in \mathbb{R}^n$ is

$$x \cdot y = \sum_{i=1}^n x_i y_i.$$

Notice that

$$|x|^2 = x \cdot x.$$

Simple inequalities, when used in a clever manner, are very powerful tools in the study of partial differential equations. We give a brief overview of some commonly used inequalities here.

The *Cauchy-Schwarz inequality* states that

$$\boxed{|x \cdot y| \leq |x||y|.}$$

To prove the Cauchy-Schwarz inequality find the value of t that minimizes

$$h(t) := |x + ty|^2.$$

For $x, y \in \mathbb{R}^n$

$$|x + y|^2 = (x + y) \cdot (x + y) = x \cdot x + x \cdot y + y \cdot x + y \cdot y.$$

Therefore

$$\boxed{|x + y|^2 = |x|^2 + 2x \cdot y + |y|^2.}$$

Using the Cauchy-Schwarz inequality we have

$$|x + y|^2 \leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2 .$$

Taking square roots of both sides we have the *triangle inequality*

$$\boxed{|x + y| \leq |x| + |y|}.$$

For $x, y \in \mathbb{R}^n$ the triangle inequality yields

$$|x| = |x - y + y| \leq |x - y| + |y|.$$

Rearranging we obtain the *reverse triangle inequality*

$$\boxed{|x-y| \geq |x| - |y|.}$$

For real numbers a, b we have

$$0 \leq (a-b)^2 = a^2 - 2ab + b^2.$$

Therefore

$$\boxed{ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2.}$$

This is called *Cauchy's inequality.*

A.3 Partial derivatives

The *partial derivative* of a function $u = u(x_1, x_2, \dots, x_n)$ in the x_i variable is defined as

$$\frac{\partial u}{\partial x_i}(x) := \lim_{h\to 0} \frac{u(x+he_i) - u(x)}{h},$$

provided the limit exists. Here $e_1, e_2, \dots, e_n$ are the standard basis vectors in $\mathbb{R}^n$, so $e_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$ has a one in the i^{th} entry. For simplicity of notation we will write

$$u_{x_i} = \frac{\partial u}{\partial x_i}.$$

The *gradient* of a function $u : \mathbb{R}^n \to \mathbb{R}$ is the vector of partial derivatives

$$\nabla u(x) := (u_{x_1}(x), u_{x_2}(x), \dots, u_{x_n}(x)).$$

We will treat the gradient as a column vector for matrix-vector multiplication.

Higher derivatives are defined iteratively. The second derivatives of u are defined as

$$\frac{\partial^2 u}{\partial x_i x_j} := \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\right).$$

This means that

$$\frac{\partial^2 u}{\partial x_i x_j}(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{\partial u}{\partial x_j}(x+he_i) - \frac{\partial u}{\partial x_j}(x)\right),$$

provided the limit exists. As before, we write

$$u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i x_j}$$

for notational simplicity. When $u_{x_i x_j}$ and $u_{x_j x_i}$ exist and are continuous we have

$$u_{x_i x_j} = u_{x_j x_i},$$

that is the second derivatives are the same regardless of which order we take them in. We will generally always assume our functions are smooth (infinitely differentiable), so equality of mixed partials is always assumed to hold.

The *Hessian* of $u : \mathbb{R}^n \to \mathbb{R}$ is the matrix of all second partial derivatives

$$\nabla^2 u(x) := (u_{x_i x_j})_{i,j=1}^n = \begin{bmatrix} u_{x_1x_1} & u_{x_1x_2} & u_{x_1x_3} & \cdots & u_{x_1x_n} \\ u_{x_2x_1} & u_{x_2x_2} & u_{x_2x_3} & \cdots & u_{x_2x_n} \\ u_{x_3x_1} & u_{x_3x_2} & u_{x_3x_3} & \cdots & u_{x_3x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{x_nx_1} & u_{x_nx_2} & u_{x_nx_3} & \cdots & u_{x_nx_n} \end{bmatrix}$$

Since we have equality of mixed partials, the Hessian is a symmetric matrix, i.e., $(\nabla^2 u)^T = \nabla^2 u$. Since we treat the gradient ∇u as a column vector, the product $\nabla^2 u(x)\nabla u(x)$ denotes the Hessian matrix multiplied by the gradient vector. That is,

$$[\nabla^2 u(x)\nabla u(x)]_j = \sum_{i=1}^n u_{x_i x_j} u_{x_i}.$$

Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ where $F(x) = (F^1(x), F^2(x), \ldots, F^n(x))$, the *divergence* of F is defined as

$$\mathrm{div} F(x) := \sum_{i=1}^n F^i_{x_i}(x).$$

The *Laplacian* of a function $u : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\Delta u := \mathrm{div}(\nabla u) = \sum_{i=1}^n u_{x_i x_i}.$$

A.4 Rules for differentiation

Most of the rules for differentiation from single variable calculus carry over to multi-variable calculus.

Chain rule: If $v(t) = (v_1(t), v_2(t), \ldots, v_n(t))$ is a function $v : \mathbb{R} \to \mathbb{R}^n$, and $u : \mathbb{R}^n \to \mathbb{R}$, then

$$\frac{d}{dt} u(v(t)) = \nabla u(v(t)) \cdot v'(t) = \sum_{i=1}^n u_{x_i}(v(t)) v_i'(t). \tag{A.1}$$

Here $v'(t) = (v_1'(t), v_2'(t), \ldots, v_n'(t))$.

If $F(x) = (F^1(x), F^2(x), \ldots, F^n(x)$ is a function $F : \mathbb{R}^n \to \mathbb{R}^n$ then

$$\frac{\partial}{\partial x_j} u(F(x)) = \nabla u(F(x)) \cdot F_{x_j}(x) = \sum_{i=1}^n u_{x_i}(F(x)) F^i_{x_j}(x),$$

where $F_{x_j} = (F^1_{x_j}, F^2_{x_j}, \ldots, F^n_{x_j})$. This is a special case of (A.1) with $t = x_j$.

Product rule: Given two functions $u, v : \mathbb{R}^n \to \mathbb{R}$, we have

$$\nabla(uv) = u\nabla v + v\nabla u.$$

This is entry-wise the usual product rule for single variable calculus.

Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ and a function $u : \mathbb{R}^n \to \mathbb{R}$ we have

$$\frac{\partial}{\partial x_i}(uF^i) = u_{x_i}F^i + uF^i_{x_i}.$$

Therefore

$$\operatorname{div}(uF) = \nabla u \cdot F + u\operatorname{div}F.$$

Exercise 6. Let $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

(a) Show that for $x \neq 0$

$$\frac{\partial}{\partial x_i}|x| = \frac{x_i}{|x|}.$$

(b) Show that for $x \neq 0$

$$\frac{\partial^2}{\partial x_i \partial x_j}|x| = \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3},$$

where δ_{ij} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

(c) Show that for $x \neq 0$

$$\Delta|x| = \frac{n-1}{|x|}.$$

Exercise 7. Find all real numbers α for which $u(x) = |x|^\alpha$ is a solution of Laplace's equation

$$\Delta u(x) = 0 \quad \text{for } x \neq 0.$$

Exercise 8. Let $1 \leq p \leq \infty$. The p-Laplacian is defined by

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

for $1 \leq p < \infty$, and

$$\Delta_\infty u := \frac{1}{|\nabla u|^2}\sum_{i=1}^n \sum_{j=1}^n u_{x_i x_j} u_{x_i} u_{x_j}.$$

Notice that $\Delta_2 u = \Delta u$. A function u is called *p-harmonic* if $\Delta_p u = 0$.

(a) Show that

$$\Delta_p u = |\nabla u|^{p-2}\left(\Delta u + (p-2)\Delta_\infty u\right).$$

(b) Show that

$$\Delta_\infty u = \lim_{p\to\infty} \frac{1}{p}|\nabla u|^{2-p}\Delta_p u.$$

Exercise 9. Let $1 \leq p \leq \infty$. Find all real numbers α for which the function $u(x) = |x|^\alpha$ is p-harmonic away from $x = 0$.

A.5 Taylor series

A.5.1 One dimension

Let $u : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Then by the fundamental theorem of calculus

$$u(y) - u(x) = \int_x^y u'(t)\, dt = \int_x^y u'(x) + u'(t) - u'(x)\, dt.$$

Since $\int_x^y u'(x)\, dt = u'(x)(y-x)$, it follows that

$$u(y) = u(x) + u'(x)(y-x) + R_2(x,y),$$

where R_2 is the remainder given by

$$R_2(x,y) = \int_x^y u'(t) - u'(x)\, dt.$$

Applying the fundamental theorem again we have

$$R_2(x,y) = \int_x^y \int_x^t u''(s)\, ds\, dt. \tag{A.2}$$

Let $C > 0$ denote the maximum value of $|u''(s)|$. Assuming, without loss of generality, that $y > x$ we have

$$|R_2(x,y)| \leq \left| \int_x^y C|t-x|\, dt \right| = \frac{C}{2}|y-x|^2.$$

Exercise 10. Verify the final equality above.

When $|g(y)| \leq C|y|^k$ we write $g \in O(|y|^k)$. Thus $R_2(x,y) \in O(|y-x|^2)$ and we have deduced the first order Taylor series

$$\boxed{u(y) = u(x) + u'(x)(y-x) + O(|y-x|^2).} \tag{A.3}$$

A Taylor series expresses the fact that a sufficiently smooth function can be well-approximated locally by its tangent line. It is important to keep in mind that the constant C hidden in the $O((y-x)^2)$ term depends on how large $|u''|$ is. Also note that we can choose $C > 0$ to be the maximum of $|u''(s)|$ for s between x and y, which may be much smaller than the maximum of $|u'(s)|$ over all s (which may not exist).

Suppose for a moment that $u''(s) \geq \theta$ for all s. Then by (A.2) we have

$$R_2(x,y) \geq \theta \int_x^y \int_x^t ds\, dt = \frac{\theta}{2}(y-x)^2.$$

provided $y > x$. If $x > y$ then

$$R_2(x,y) = \int_y^x \int_t^x u''(s)\, ds\, dt \geq \theta \int_x^y \int_x^t ds\, dt = \frac{\theta}{2}(y-x)^2.$$

Either way we have

$$\boxed{u(y) \geq u(x) + u'(x)(y-x) + \frac{\theta}{2}(y-x)^2.} \tag{A.4}$$

It is useful sometimes to continue the Taylor series to higher order terms. For this, suppose u is three times continuously differentiable. We first write the Taylor series with remainder for $u'(t)$

$$u'(t) = u'(x) + u''(x)(t-x) + \int_x^t \int_x^\tau u'''(s)\,ds\,d\tau.$$

Proceeding as before, we use the fundamental theorem of calculus to find

$$\begin{aligned}
u(y) &= u(x) + \int_x^y u'(t)\,dt \\
&= u(x) + \int_x^y u'(x) + u''(x)(t-x) + \int_x^t \int_x^\tau u'''(s)\,ds\,d\tau\,dt \\
&= u(x) + u'(x)(y-x) + \frac{1}{2}u''(x)(y-x)^2 + R_3(x,y),
\end{aligned}$$

where

$$R_3(x,y) = \int_x^y \int_x^t \int_x^\tau u'''(s)\,ds\,d\tau\,dt.$$

As before, let $C > 0$ denote the maximum value of $|u'''(s)|$. Then

$$|R_3(x,y)| \leq \frac{C}{6}|y-x|^3.$$

Exercise 11. Verify the inequality above.

Therefore $R_3 \in O(|y-x|^3)$ and we have the second order Taylor expansion

$$\boxed{u(y) = u(x) + u'(x)(y-x) + \frac{1}{2}u''(x)(y-x)^2 + O(|y-x|^3).} \tag{A.5}$$

The second order Taylor series says that a sufficiently smooth function can be approximated up to $O((y-x)^3)$ accuracy with a parabola. Again, we note that the constant $C > 0$ hidden in $O((y-x)^3)$ depends on the size of $|u'''(s)|$, and $C > 0$ may be chosen as the maximum of $|u'''(s)|$ over s between x and y.

A.5.2 Higher dimensions

Taylor series expansions for functions $u : \mathbb{R}^n \to \mathbb{R}$ follow directly from the one dimensional case and the chain rule. Suppose u is twice continuously differentiable and fix $x, y \in \mathbb{R}^n$. For $t \in \mathbb{R}$ define

$$\varphi(t) = u(x + (y-x)t).$$

Since φ is a function of one variable t, we can use the one dimensional Taylor series to obtain

$$\varphi(t) = \varphi(0) + \varphi'(0)t + O(|t|^2). \tag{A.6}$$

The constant in the $O(|t|^2)$ term depends on the maximum of $|\varphi''(t)|$. All that remains is to compute the derivatives of φ. By the chain rule

$$\varphi'(t) = \sum_{i=1}^n u_{x_i}(x + (y-x)t)(y_i - x_i), \tag{A.7}$$

and

$$\begin{aligned}
\varphi''(t) &= \frac{d}{dt}\sum_{i=1}^{n} u_{x_i}(x+(y-x)t)(y_i-x_i) \\
&= \sum_{i=1}^{n} \frac{d}{dt} u_{x_i}(x+(y-x)t)(y_i-x_i) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_ix_j}(x+(y-x)t)(y_i-x_i)(y_j-x_j). \qquad \text{(A.8)}
\end{aligned}$$

In particular

$$\varphi'(0) = \sum_{i=1}^{n} u_{x_i}(x)(y_i-x_i) = \nabla u(x)\cdot(y-x),$$

and so (A.6) with $t=1$ becomes

$$u(y) = u(x) + \nabla u(x)\cdot(y-x) + R_2(x,y),$$

where $R_2(x,y)$ satisfies $|R_2(x,y)| \leq \frac{1}{2}\max_t|\varphi''(t)|$. Let $C>0$ denote the maximum value of $|u_{x_ix_j}(z)|$ over all z, i and j. Then by (A.8)

$$|\varphi''(t)| \leq C\sum_{i=1}^{n}\sum_{j=1}^{n}|y_i-x_i||y_j-x_j| \leq Cn^2|x-y|^2.$$

It follows that $|R_2(x,y)| \leq \frac{C}{2}n^2|x-y|^2$, hence $R_2(x,y)\in O(|x-y|^2)$ and we arrive at the first order Taylor series

$$\boxed{u(y) = u(x) + \nabla u(x)\cdot(y-x) + O(|x-y|^2).} \qquad \text{(A.9)}$$

This says that u can be locally approximated near x to order $O(|x-y|^2)$ by the affine function

$$L(y) = u(x) + \nabla u(x)\cdot(y-x).$$

We can continue this way to obtain the second order Taylor expansion. We assume now that u is three times continuously differentiable. Using the one dimensional second order Taylor expansion we have

$$\varphi(t) = \varphi(0) + \varphi'(0)t + \frac{1}{2}\varphi''(0)t^2 + O(|t|^3). \qquad \text{(A.10)}$$

The constant in the $O(|t|^3)$ term depends on the maximum of $|\varphi'''(t)|$. Notice also that

$$\varphi''(0) = \sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_ix_j}(x)(y_i-x_i)(y_j-x_j) = (y-x)\cdot\nabla^2u(x)(y-x),$$

where $\nabla^2u(x)$ is the Hessian matrix. Plugging this into (A.10) with $t=1$ yields

$$u(y) = u(x) + \nabla u(x)\cdot(y-x) + \frac{1}{2}(y-x)\cdot\nabla^2u(x)(y-x) + R_3(x,y),$$

where $R_3(x,y)$ satisfies $|R_3(x,y)| \leq \frac{1}{6}\max_t |\varphi'''(t)|$. We compute

$$\begin{aligned}\varphi'''(t) &= \sum_{i=1}^n \sum_{j=1}^n \frac{d}{dt} u_{x_ix_j}(x+(y-x)t)(y_i-x_i)(y_j-x_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n u_{x_ix_jx_k}(x+(y-x)t)(y_i-x_i)(y_j-x_j)(y_k-x_k).\end{aligned}$$

Let $C>0$ denote the maximum value of $|u_{x_ix_jx_k}(z)|$ over all z, i, j, and k. Then we have

$$|\varphi'''(t)| \leq C\sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n |y_i-x_i||y_j-x_j||y_k-x_k| \leq Cn^3|x-y|^3.$$

Therefore $|R_3(x,y)| \leq \frac{C}{6}n^3|x-y|^3$ and so $R_3 \in O(|x-y|^3)$. Finally we arrive at the second order Taylor expansion

$$\boxed{u(y) = u(x) + \nabla u(x)\cdot(y-x) + \frac{1}{2}(y-x)\cdot\nabla^2 u(x)(y-x) + O(|x-y|^3).} \tag{A.11}$$

This says that u can be locally approximated near x to order $O(|x-y|^3)$ by the quadratic function

$$L(y) = u(x) + \nabla u(x)\cdot(y-x) + \frac{1}{2}(y-x)\cdot\nabla^2 u(x)(y-x).$$

Notice that by (A.8), the second derivative of u at x in the direction $v \in \mathbb{R}^n$ is given by

$$\frac{d^2}{dt^2}u(x+tv) = \sum_{i=1}^n \sum_{j=1}^n u_{x_ix_j}(x)v_iv_j.$$

Let us suppose that all the second derivatives of u are lower bounded by θ, that is we assume

$$\boxed{\sum_{i=1}^n \sum_{j=1}^n u_{x_ix_j}(x)v_iv_j \geq \theta|v|^2 \quad \text{for all } x, v \in \mathbb{R}^n.} \tag{A.12}$$

Now set $\varphi(t) = u(x+tv)$ and note that

$$\varphi'(t) = \sum_{i=1}^n u_{x_i}(x+tv)v_i = \nabla u(x+tv)\cdot v,$$

and

$$\varphi''(t) = \sum_{i=1}^n \sum_{j=1}^n u_{x_ix_j}(x+tv)v_iv_j \geq \theta|v|^2. \tag{A.13}$$

Therefore by (A.4) we have

$$\varphi(t) \geq \varphi(0) + \varphi'(0)t + \frac{\theta}{2}|v|^2t^2.$$

Substituting the expressions above we have

$$u(x+tv) \geq u(x) + \nabla u(x) \cdot v + \frac{\theta}{2}|v|^2 t^2.$$

Finally we set $v = y - x$ and $t = 1$ to obtain

$$\boxed{u(y) \geq u(x) + \nabla u(x) \cdot (y-x) + \frac{\theta}{2}|y-x|^2.} \tag{A.14}$$

A.6 Topology

We will have to make use of basic point-set topology. We define the open ball of radius $r > 0$ centered at $x_0 \in \mathbb{R}^n$ by

$$B^0(x_0, r) := \{x \in \mathbb{R}^n : |x - x_0| < r\}.$$

The closed ball is defined as

$$B(x_0, r) := \{x \in \mathbb{R}^n : |x - x_0| \leq r\}.$$

Definition 3. A set $U \subset \mathbb{R}^n$ is called *open* if for each $x \in U$ there exists $r > 0$ such that $B(x, r) \subset U$.

Exercise 12. Let $U, V \subset \mathbb{R}^n$ be open. Show that

$$W := U \cup V := \{x \in \mathbb{R}^n : x \in U \text{ or } x \in V\}$$

is open.

Definition 4. We say that a sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$ *converges* to $x \in \mathbb{R}^n$, written $x_k \to x$, if

$$\lim_{k \to \infty} |x_k - x| = 0.$$

Definition 5. The *closure* of a set $U \subset \mathbb{R}^n$, denotes $\overline{U}$, is defined as

$$\overline{U} := \{x \in \mathbb{R}^n : \text{there exists a sequence } x_k \in U \text{ such that } x_k \to x\}.$$

The closure is the set of points that can be reached as limits of sequences belonging to U.

Definition 6. We say that a set $U \subset \mathbb{R}^n$ is *closed* if $\overline{U} = U$.

Exercise 13. Another definition of *closed* is: A set $U \subset \mathbb{R}^n$ is *closed* if the complement

$$\mathbb{R}^n \setminus U := \{x \in \mathbb{R}^n : x \notin U\}$$

is open. Verify that the two definitions are equivalent [This is not a trivial exercise].

Definition 7. We define the *boundary* of an open set $U \subset \mathbb{R}^n$, denoted ∂U, as

$$\partial U := \overline{U} \setminus U.$$

Example 11. The open ball $B^0(x,r)$ is open, and its closure is the closed ball $B(x,r)$. The boundary of the open ball $B^0(x,r)$ is

$$\partial B^0(x_0,r) := \{x \in \mathbb{R}^n \,:\, |x - x_0| = r\}.$$

It is a good idea to verify each of these facts from the definitions.

We defined the boundary only for open sets, but is can be defined for any set.

Definition 8. The *interior* of a set $U \subset \mathbb{R}^n$, denoted $\mathrm{int}(U)$, is defined as

$$\mathrm{int}(U) := \{x \in U \,:\, B(x,r) \subset U \text{ for small enough } r > 0\}.$$

Exercise 14. Show that $U \subset \mathbb{R}^n$ is open if and only if $\mathrm{int}(U) = U$.

We can now define the boundary of an arbitrary set $U \subset \mathbb{R}^n$.

Definition 9. We define the *boundary* of a set $U \subset \mathbb{R}^n$, denoted ∂U, as

$$\partial U := \overline{U} \setminus \mathrm{int}(U).$$

Exercise 15. Verify that

$$\partial B(x,r) = \partial B^0(x,r).$$

Definition 10. We say a set $U \subset \mathbb{R}^n$ is *bounded* if there exists $M > 0$ such that $|x| \leq M$ for all $x \in U$.

Definition 11. We say a set $U \subset \mathbb{R}^n$ is *compact* if U is closed and bounded.

Definition 12. For open sets $V \subset U \subset \mathbb{R}^n$ we say that V is *compactly contained* in U if $\overline{V}$ is compact and $\overline{V} \subset U$. If V is compactly contained in U we write $V \subset\subset U$.

A.7 Function spaces

For an open set $U \subset \mathbb{R}^n$ we define

$$C^k(\overline{U}) := \big\{\text{Functions } u : \overline{U} \to \mathbb{R} \text{ that are } k\text{-times continuously differentiable on } \overline{U}\big\}.$$

The terminology k-times continuously differentiable means that all k^{th}-order partial derivatives of u exist and are continuous on $\overline{U}$. We write $C^0(\overline{U}) = C(\overline{U})$ for the space of functions that are continuous on $\overline{U}$.

Exercise 16. Show that the function $u(x) = x^2$ for $x > 0$ and $u(x) = -x^2$ for $x \leq 0$ belongs to $C^1(\mathbb{R})$ but not to $C^2(\mathbb{R})$.

We also define

$$C^\infty(\overline{U}) := \bigcap_{k=1}^{\infty} C^k(\overline{U})$$

to be the space of infinitely differentiable functions. Functions $u \in C^\infty(\overline{U})$ are called *smooth.*

Definition 13. The *support* of a function $u : \overline{U} \to \mathbb{R}$ is defined as

$$\operatorname{supp}(u) := \{x \in \overline{U} \,:\, u(x) \neq 0\}.$$

Definition 14. We say that $u : \overline{U} \to \mathbb{R}$ is *compactly supported* in U if $\operatorname{supp}(u) \subset\subset U$.

A function u is compactly supported in U if u vanishes near the boundary ∂U. Finally for $k \in \mathbb{N} \cup \{\infty\}$ we write

$$C_c^k(U) := \{u \in C^k(\overline{U}) \,:\, u \text{ is compactly supported in } U\}.$$

For a function $u : U \to \mathbb{R}$ we define the L^2-norm of u to be

$$\|u\|_{L^2(U)} := \left(\int_U u^2 \, dx\right)^{\frac{1}{2}}.$$

For two functions $u, v : U \to \mathbb{R}$ we define the L^2-inner product of u and v to be

$$\langle u, v\rangle_{L^2(U)} := \int_U u \, v \, dx.$$

Notice that

$$\|u\|_{L^2(U)}^2 = \langle u, u\rangle_{L^2(U)}.$$

We also define

$$L^2(U) := \Big\{\text{Functions } u : U \to \mathbb{R} \text{ for which } \|u\|_{L^2(U)} < \infty\Big\}.$$

$L^2(U)$ is a Hilbert space (a complete inner-product space). We will often write $\|u\|$ in place of $\|u\|_{L^2(U)}$ and $\langle u, v\rangle$ in place of $\langle u, v\rangle_{L^2(U)}$ when it is clear from the context that the L^2 norm is intended.

As before, we have the Cauchy-Schwarz inequality

$$\langle u, v\rangle_{L^2(U)} \leq \|u\|_{L^2(U)} \|v\|_{L^2(U)}.$$

We also have

$$\|u + v\|_{L^2(U)}^2 = \|u\|_{L^2(U)}^2 + 2\langle u, v\rangle_{L^2(U)} + \|v\|_{L^2(U)}^2.$$

Applying the Cauchy-Schwarz inequality we get the *triangle inequality*

$$\|u + v\|_{L^2(U)} \leq \|u\|_{L^2(U)} + \|v\|_{L^2(U)},$$

and the *reverse triangle inequality*

$$\|u - v\|_{L^2(U)} \geq \|u\|_{L^2(U)} - \|v\|_{L^2(U)}.$$

A.8 Integration by parts

All of the sets $U \subset \mathbb{R}^n$ that we work with will be assumed to be open and bounded with a smooth boundary ∂U. A set $U \subset \mathbb{R}^n$ has a smooth boundary if at each point $x \in \partial U$ we can make an orthogonal change of coordinates so that for some $r > 0$, $\partial U \cap B(0, r)$ is the graph of a smooth function $u : \mathbb{R}^{n-1} \to \mathbb{R}$. If ∂U is smooth, we can define an outward normal vector $\nu = \nu(x)$ at each point $x \in \partial U$, and ν varies smoothly with x. Here, $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n$ and ν is a unit vector so

$$|\nu| = \sqrt{\nu_1^2 + \cdots + \nu_n^2} = 1.$$

The *normal derivative* of $u \in C^1(\overline{U})$ at $x \in \partial U$ is

$$\frac{\partial u}{\partial \nu}(x) := \nabla u(x) \cdot \nu(x).$$

Integration by parts in $\mathbb{R}^n$ is based on the Gauss-Green Theorem.

Theorem 6 (Gauss-Green Theorem)**.** *Let $U \subset \mathbb{R}^n$ be an open and bounded set with a smooth boundary ∂U. If $u \in C^1(\overline{U})$ then*

$$\int_U u_{x_i}\, dx = \int_{\partial U} u\nu_i \, dS.$$

The Gauss-Green Theorem is the natural extension of the fundamental theorem of calculus to dimensions $n \geq 2$. A proof of the Gauss-Green Theorem is outside the scope of this course.

We can derive a great many important integration by parts formulas from the Gauss-Green Theorem. These identities are often referred to as Green's identities or simply integration by parts.

Theorem 7 (Integration by parts)**.** *Let $U \subset \mathbb{R}^n$ be an open and bounded set with a smooth boundary ∂U. If $u, v \in C^2(\overline{U})$ then*

(i) $\displaystyle\int_U u\Delta v\, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu}\, dS - \int_U \nabla u \cdot \nabla v\, dx,$

(ii) $\displaystyle\int_U u\Delta v - v\Delta u\, dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\, dS,$ *and*

(iii) $\displaystyle\int_U \Delta v\, dx = \int_{\partial U} \frac{\partial v}{\partial \nu}\, dS.$

Proof. (i) Notice that

$$\partial_{x_i}(uv_{x_i}) = u_{x_i}v_{x_i} + uv_{x_ix_i}.$$

Applying the Gauss-Green Theorem to uv_{x_i} we have

$$\int_{\partial U} uv_{x_i}\nu_i\, dS = \int_U u_{x_i}v_{x_i} + uv_{x_ix_i}\, dx.$$

Summing over i we have

$$\int_{\partial U} u\frac{\partial v}{\partial \nu}\, dS = \int_U \nabla u \cdot \nabla v + u\Delta v\, dx,$$

which is equivalent to (i).

(ii) Swap the roles of u and v in (i) and subtract the resulting identities to prove (ii).

(iii) Take $u = 1$ in (i). □

It will also be useful to prove the following version of the divergence theorem. Recall that for a vector field $F(x) = (F^1(x), \ldots, F^n(x))$ the divergence of F is

$$\mathrm{div}(F) = F^1_{x_1} + F^2_{x_2} + \cdots + F^n_{x_n}.$$

Theorem 8 (Divergence theorem)**.** *Let $U \subset \mathbb{R}^n$ be an open and bounded set with a smooth boundary ∂U. If $u \in C^1(\overline{U})$ and F is a C^1 vector field (i.e., $F^i \in C^1(\overline{U})$ for all i) then*

$$\int_U u\, div(F)\, dx = \int_{\partial U} u\, F \cdot \nu\, dS - \int_U \nabla u \cdot F\, dx.$$

Proof. The proof is similar to Theorem 7 (i). Notice that

$$(uF^i)_{x_i} = u_{x_i} F^i + u F^i_{x_i},$$

and apply the Gauss-Green Theorem to find that

$$\int_{\partial U} u F^i \nu_i\, dS = \int_U u_{x_i} F^i + u F^i_{x_i}\, dx.$$

Summing over i we have

$$\int_{\partial U} u\, F \cdot \nu\, dS = \int_U \nabla u \cdot F + u\, \mathrm{div}(F)\, dx,$$

which is equivalent to the desired result. □

Notice that when $u = 1$ Theorem 8 reduces to

$$\int_U \mathrm{div}(F)\, dx = \int_{\partial U} F \cdot \nu\, dS,$$

which is the usual divergence theorem. If we take $F = \nabla v$ for $v \in C^2(\overline{U})$, then we recover Theorem 7 (i).

Exercise 17. Let $u, w \in C^2(\overline{U})$ where $U \subset \mathbb{R}^n$ is open and bounded. Show that for $1 \le p < \infty$

$$\int_U u\, \Delta_p w\, dx = \int_{\partial U} u\, |\nabla w|^{p-2} \frac{\partial w}{\partial \nu}\, dS - \int_U |\nabla w|^{p-2} \nabla u \cdot \nabla w\, dx.$$

The p-Laplacian Δ_p was defined in Exercise 8.

A.9 Vanishing lemma

Lemma 2. *Let $U \subset \mathbb{R}^n$ be open and bounded and let $u \in C(U)$. If*

$$\int_U u(x)\varphi(x)\, dx = 0 \quad \textit{for all } \varphi \in C^\infty_c(U)$$

then $u(x) = 0$ for all $x \in U$.

Proof. Let us sketch the proof. Assume to the contrary that $u(x_0) \neq 0$ at some $x_0 \in U$. We may assume, without loss of generality that $\varepsilon := u(x_0) > 0$. Since u is continuous, there exists $\delta > 0$ such that

$$u(x) \geq \frac{\varepsilon}{2} \quad \text{whenever } |x - x_0| < \delta.$$

Now let $\varphi \in C_c^\infty(U)$ be a test function satisfying $\varphi(x) > 0$ for $|x - x_0| < \delta$ and $\varphi(x) = 0$ for $|x - x_0| \geq \delta$. Then

$$0 = \int_U u(x)\varphi(x)\,dx = \int_{B(x_0,\delta)} u(x)\varphi(x)\,dx \geq \frac{\varepsilon}{2}\int_{B(x_0,\delta)} \varphi(x)\,dx > 0,$$

which is a contradiction. $\square$

A test function satisfying the requirements in the proof of Lemma 2 is given by

$$\varphi(x) = \begin{cases} \exp\left(-\frac{1}{\delta^2 - |x-x_0|^2}\right), & \text{if } |x - x_0| < \delta \\ 0, & \text{if } |x - x_0| \geq \delta. \end{cases}$$

In Math 5587 Homework 11 Problem 5 we proved that a similar function in $n = 1$ dimensions was smooth.

A.10 Total variation of characteristic function is perimeter

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and suppose the zero level set

$$C = \{x \in \mathbb{R}^2 \,:\, u(x) = 0\}$$

is a smooth simple closed curve. We give here a short informal proof that the length of C, denote $L(u)$, is given by

$$L(u) = \int_{\mathbb{R}^2} |\nabla H(u(x))|\,dx = \int_{\mathbb{R}^2} \delta(u(x))|\nabla u(x)|\,dx, \tag{A.15}$$

where $H : \mathbb{R} \to \mathbb{R}$ is the Heaviside function defined by

$$H(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0. \end{cases}$$

We note that the second equality in (A.15) is just the chain rule.

Since $L(u)$ is the length of the zero level set of u, the value $L(u)$ depends only on the zero level set. Thus, if $v : \mathbb{R}^2 \to \mathbb{R}$ is any other smooth function for which

$$\{x \in \mathbb{R}^2 \,:\, u(x) = 0\} = C = \{x \in \mathbb{R}^2 \,:\, v(x) = 0\} \tag{A.16}$$

then $L(u) = L(v)$. If we also assume that

$$U := \{x \in \mathbb{R}^2 \,:\, u(x) > 0\} = \{x \in \mathbb{R}^2 \,:\, v(x) > 0\} \tag{A.17}$$

then $H(u(x)) = H(v(x))$ and hence the right hand side of (A.15) is identical for u and v. Hence, instead of proving (A.15) for u, we may prove it for any v satisfying (A.16) and (A.17).

Let us choose v to be the signed distance function to the zero level set C of u. That is, we define

$$v(x) = \begin{cases} \mathrm{dist}(x, C), & \text{if } x \in U \\ -\mathrm{dist}(x, C), & \text{if } x \notin U, \end{cases} \tag{A.18}$$

where

$$\mathrm{dist}(x, C) = \min_{y \in C} |x - y|.$$

Then (A.16) and (A.17) hold, and $|\nabla v(x)| = 1$.[1] To see why $|\nabla v| = 1$, recall that

$$|\nabla v(x)| = \nabla v(x) \cdot \left(\frac{\nabla v(x)}{|\nabla v(x)|} \right) = \frac{d}{dt}\Big|_{t=0} v(x + tp),$$

where $p = \frac{\nabla v(x)}{|\nabla v(x)|}$. So $|\nabla v(x)|$ is the rate of change of v in the direction of the gradient, and the gradient is the direction of steepest ascent. For the distance function (A.18) the greatest rate of change is 1, since v is the distance to C.

Now let us define

$$A(t) = \int_{\mathbb{R}^2} H(v(x) + t)\, dx$$

for $t \in \mathbb{R}$. The quantity $A(t)$ represents the area of the set $U(t) = \{x \in \mathbb{R}^2 : v(x) > -t\}$. We may assume, without loss of generality, that U is the interior of C. Then for $t > 0$, $U \subset U(t)$, and since $|\nabla v| = 1$ the set $U(t)$ is larger than U by a distance of t in the normal direction to the boundary. Hence

$$A(t) - A(0) \approx L(v)t = L(u)t.$$

Dividing by t and sending t to zero we have

$$L(u) = A'(0) = \int_{\mathbb{R}^2} \delta(v(x))\, dx = \int_{\mathbb{R}^2} |\nabla H(v)|\, dx = \int_{\mathbb{R}^2} |\nabla H(u)|\, dx.$$

This establishes the claim.

References

[1] T. F. Chan and L. A. Vese. Active contours without edges. *IEEE Transactions on image processing*, 10(2):266–277, 2001.

[2] L. I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D: Nonlinear Phenomena*, 60(1):259–268, 1992.

[1] Actually, there will be points where ∇v does not exist, but at least in a neighborhood of C we are guaranteed that ∇v exists and $|\nabla v| = 1$.