Math 5588 Final Exam Information

- The final exam will take place on Tuesday, May 9 4:45pm-6:45pm in Vincent Hall 206 (Please note the room number!). Please arrive early so we can start on time.
- The exam can potentially cover anything in the course, with the exception of the lectures on differential games.
- The exam is closed book. No textbooks, notes, or calculators are allowed. The formula sheet below will be provided on the exam.
- The exam will have 8 questions, ranging in levels of difficulty. It is a good idea to look through the questions first and complete the ones your are most comfortable with early in the exam. Below are a collection of sample questions from the last third of the course. Please see the sample problems for midterms 1 and 2 for practice problems from the first two thirds of the semester.

Sample questions

1. Determine whether the following statements are true or false. No justification is required.
   (a) Disturbances in the wave equation travel at exactly the wavespeed c in $n = 3$ dimensions.
   (b) Disturbances in the wave equation travel at exactly the wavespeed c in $n = 2$ dimensions.
   (c) The Rankine-Hugoniot condition selects a unique weak solution of a scalar conservation law.
   (d) If $u(x,t)$ is a solution of a Hamilton-Jacobi equation in $n = 1$ spatial dimensions, then $v = u_x$ is the solution of a scalar conservation law.
   (e) The Hopf-Lax formula gives the unique viscosity solution a Hamilton-Jacobi equation of the form $u_t + H(\nabla u) = 0$ when H is convex and superlinear.
   (f) The Lax-Oleinik formula is an alternative to the Hopf-Lax formula for Hamilton-Jacobi equations.

2. Use the Hopf-Lax formula to solve the Hamilton-Jacobi equation

$$u_t + \frac{1}{2}u_x^2 = 0$$

   with initial condition $u(x,0) = f(x)$ where
   (a) $f(x) = x$
   (b) $f(x) = x^2$
   (c) $f(x) = 1 - x$

3. Solve Burger's equation $u_t + uu_x = 0$ with initial condition $u(x,0) = f(x)$ where
   (a) $f(x) = x$

(b) $f(x) = x^2$

(c) $f(x) = 1/x$

4. Find the entropy solution of Burger's equation $u_t + uu_x = 0$ with initial condition $u(x, 0) = f(x)$ where

$$f(x) = \begin{cases} 1, & \text{if } x < 0 \\ 0, & \text{if } 0 < x < 1 \\ 1, & \text{if } x > 1. \end{cases}$$

5. Find the entropy solution of Burger's equation $u_t + uu_x = 0$ with initial condition $u(x, 0) = f(x)$ where

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } 0 < x < 1 \\ 0, & \text{if } x > 1. \end{cases}$$

6. Calculate the elemental stiffness k^s_{ij} for an isosceles triangle with base of length 1 and height $h > 0$.

7. Consider the optimal control problem with dynamics

$$\dot{x}(s) = \alpha(s) - x(s) + b, \quad t < s \leq T$$

and cost

$$C_{x,t}(\alpha) = \int_t^T |\alpha(s)|^2 + |x(s)|^2 \, ds,$$

where $b \in \mathbb{R}^n$, $\alpha : [0, T] \to \mathbb{R}^n$ is the control and $x : [0, T] \to \mathbb{R}^n$ is the state. Write down the Hamilton-Jacobi-Bellman equation for the value function

$$u(x, t) = \min_{\alpha:[0,T]\to\mathbb{R}^n} C_{x,t}(\alpha).$$

Formula Sheet

$$L(u(x),u'(x)) - u'(x)L_p(u(x),u'(x)) = \text{Constant}$$

$$L_z(x,u(x),u'(x)) - \frac{d}{dx}L_p(x,u(x),u'(x)) = 0.$$

$$\nabla I(u) = L_z(x,u,\nabla u) - \operatorname{div}\left(\nabla_p L(x,u,\nabla u)\right) = 0$$

$$\int_U u_{x_i}\,dx = \int_{\partial U} u\nu_i\,dS.$$

$$\int_U u\Delta v\,dx = \int_{\partial U} u\frac{\partial v}{\partial \nu}\,dS - \int_U \nabla u\cdot\nabla v\,dx$$

$$\int_U u\Delta v - v\Delta u\,dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\,dS$$

$$\int_U \Delta v\,dx = \int_{\partial U} \frac{\partial v}{\partial \nu}\,dS$$

$$\int_U u\operatorname{div}(F)\,dx = \int_{\partial U} u\,F\cdot\nu\,dS - \int_U \nabla u\cdot F\,dx.$$

$$\mathcal{F}(u) = \widehat{u}(k) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} u(x)e^{-ik\cdot x}\,dx.$$

$$\mathcal{F}^{-1}(\widehat{u}) = u(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \widehat{u}(k)e^{ik\cdot x}\,dk.$$

$$\mathcal{F}(u_{x_j}) = ik_j\widehat{u}(k).$$

$$\mathcal{F}(u*v) = (2\pi)^{n/2}\widehat{u}(k)\widehat{v}(k).$$

$$\int_{\mathbb{R}^n} |\widehat{u}(k)|^2\,dk = \int_{\mathbb{R}^n} |u(x)|^2\,dx.$$

$$\mathcal{F}(e^{-|x|^2/2}) = e^{-|k|^2/2}.$$

$$u(x,t) = \frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} tg(y) + f(y) + \nabla f(y)\cdot(y-x)\,dS(y).$$

$$u(x,t) = \frac{1}{2\pi t^2}\int_{B(x,t)} \frac{tf(y) + t\nabla f(y)\cdot(y-x) + t^2 g(y)}{\sqrt{t^2 - |x-y|^2}}\,dy.$$

$$k_{ij}^s = -\frac{(\mathbf{x}_i - \mathbf{x}_\ell)\cdot(\mathbf{x}_j - \mathbf{x}_\ell)}{4\text{Area}(T_s)}, \quad k_{ii}^s = \frac{|\mathbf{x}_j - \mathbf{x}_\ell|^2}{4\text{Area}(T_s)}.$$

$$\frac{dx}{dt} = \frac{F(u_\ell) - F(u_R)}{u_\ell - u_R}, \quad u_\ell > u_R.$$

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left\{f(y) + tL\left(\frac{x-y}{t}\right)\right\}, \quad L = H^*$$

$$H^*(q) = \max_{p\in\mathbb{R}^n}\{p\cdot q - H(p)\}.$$

$$H(p,x) = \min_{a\in A}\{f(x,a)\cdot p + r(x,a)\}.$$