Math 5588 – Homework 11 (Due Thursday April 20)

1. A weak solution of Burger's equation

$$u_t + uu_x = 0 \quad \text{for } x \in \mathbb{R},\ t > 0$$

has the form

$$u(x,t) = \begin{cases} \frac{x}{t+1}, & \text{for } x < s(t) \\ 0, & \text{for } x > s(t), \end{cases}$$

where $s(t)$ is a shock curve starting at $s(0) = 1$. Find a formula for $s(t)$. [Hint: Use the Rankine-Hugoniot condition to find an ODE that $s(t)$ satisfies.]

2. Find the solution of Riemann's problem, which is Burger's equation

$$u_t + uu_x = 0$$

with initial condition $u(x,0) = f(x)$ where

$$f(x) = \begin{cases} u_l, & \text{if } x < 0 \\ u_r, & \text{if } x > 0, \end{cases}$$

where u_l and u_r are constants describing the initial states to the left and right of zero, respectively. You can assume $u_l \neq u_r$. [Hint: Depending on whether $u_l < u_r$ or $u_l > u_r$ you will have either a rarefaction wave or a shock wave. Give the solution in both cases.]

3. Recall the Legendre transform of a function $H(p)$ is given by

$$H^*(q) = \max_{p \in \mathbb{R}} \{pq - H(p)\}.$$

Find the Legendre transform of $H(p) = \frac{1}{2}p^2$ and show directly that $H^{**} = H$. [Here, $H^{**} = (H^*)^*$.]

4. Show that the Legendre transform of the function $H(p) = \frac{1}{r}|p|^r$ for $1 < r < \infty$ is

$$H^*(q) = \frac{1}{s}|q|^s$$

where $\frac{1}{s} + \frac{1}{r} = 1$.