MATH 5588 – HOMEWORK 2 (DUE THURSDAY JANUARY 26)

In all problems $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and u is a real-valued function on $\mathbb{R}^n$, $u : \mathbb{R}^n \to \mathbb{R}$. This aim of this homework is to give you practice with multi-variable calculus.

1. Let $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.
   (a) Show that for $x \neq 0$
   $$\frac{\partial}{\partial x_i}|x| = \frac{x_i}{|x|}.$$
   (b) Show that for $x \neq 0$
   $$\frac{\partial^2}{\partial x_i \partial x_j}|x| = \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3},$$
   where δ_{ij} is the Kronecker delta defined by
   $$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$
   (c) Show that for $x \neq 0$
   $$\Delta|x| = \frac{n-1}{|x|}.$$
2. Find all real numbers α for which $u(x) = |x|^\alpha$ is a solution of Laplace's equation
   $$\Delta u(x) = 0 \quad \text{for } x \neq 0.$$
3. Let $1 \leq p \leq \infty$. The p-Laplacian is defined by
   $$\Delta_p u := \text{div}\left(|\nabla u|^{p-2}\nabla u\right)$$
   for $1 \leq p < \infty$, and
   $$\Delta_\infty u := \frac{1}{|\nabla u|^2}\sum_{i=1}^n \sum_{j=1}^n u_{x_i x_j} u_{x_i} u_{x_j}.$$
   Notice that $\Delta_2 u = \Delta u$. A function u is called *p-harmonic* if $\Delta_p u = 0$.
   (a) Show that
   $$\Delta_p u = |\nabla u|^{p-2}\left(\Delta u + (p-2)\Delta_\infty u\right).$$
   (b) Show that
   $$\Delta_\infty u = \lim_{p\to\infty} \frac{1}{p}|\nabla u|^{2-p}\Delta_p u.$$
4. Let $1 \leq p \leq \infty$. Find all real numbers α for which the function $u(x) = |x|^\alpha$ is p-harmonic away from $x = 0$.
5. Let $u, w \in C^2(\overline{U})$ where $U \subseteq \mathbb{R}^n$ is open and bounded. Show that for $1 \leq p < \infty$
   $$\int_U u\,\Delta_p w\,dx = \int_{\partial U} u\,|\nabla w|^{p-2}\frac{\partial w}{\partial \nu}\,dS - \int_U |\nabla w|^{p-2}\nabla u \cdot \nabla w\,dx.$$
   [Hint: Use one of the integration by parts formulas in the appendix of the course notes.]