Math 5588 – Homework 3 (Due Thursday February 2)

1. Find and solve the Euler-Lagrange equation for the functional

$$I(u) = \int_0^{\log(2)} u(x)^2 + (u'(x))^2 \, dx$$

subject to boundary conditions $u(0) = 0$ and $u(\log(2)) = 1$, where log is the natural logarithm. Do you think your solution is a minimum or a maximum (or neither)?

2. Find the Euler-Lagrange equation for the functional

$$I(u) = \int_0^{\pi} u(x)^2 - (u'(x))^2 \, dx$$

subject to boundary conditions $u(0) = u(\pi) = 0$. Then find *all* solutions of the Euler-Lagrange equation (as a one-parameter family) and evaluate I on all solutions.

3. Find the Euler-Lagrange equation for the functional

$$I(u) = \int_U (f(x) - u(x))^2 + \lambda\varphi(|\nabla u(x)|) \, dx$$

where $f : U \to \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$, and λ is a constant.

4. Show that the Euler-Lagrange equation for the functional

$$I(u) = \int_U \frac{1}{p}|\nabla u(x)|^p - u(x)f(x) \, dx$$

is the p-Laplace equation

$$-\Delta_p u = f \quad \text{in } U.$$

[Hint: The p-Laplacian was defined in Homework 2.]

5. Find the Euler-Lagrange equation for the functional

$$I(u) = \int_U (\Delta u(x))^2 \, dx.$$

[Hint: Proceed as in the proof of the Euler-Lagrange equation from class. That is, let φ be smooth with compact support in U and compute

$$\left.\frac{d}{dt}\right|_{t=0} I(u + t\varphi) = 0.$$

Use integration by parts and the vanishing lemma to find the Euler-Lagrange equation.]