# Math 5588 – Homework 4 (Due Thursday February 9)

1. Consider the following version of the isoperimetric problem:

$$\max_{\substack{u:[-1,1]\to\mathbb{R}\\ u(-1)=0=u(1)}} \int_{-1}^{1} u(x)\,dx \ \text{ subject to } \int_{-1}^{1} \sqrt{1+u'(x)^2}\,dx = L,$$

where $L > 2$. Show that the optimal curve $C(x) = (x, u(x))$ must be a segment of a circle. [Hint: Proceed in a similar fashion to the Brachistochrone problem. In particular, after reducing the Euler-Lagrange equation to a first order ODE, you should solve it by reparamaterizing the curve $C$ in terms of the angle $\theta$ between the tangent vector to $C$ and the $x$-axis.]

2. Consider the constrained problem

$$\min_{\substack{u:U\to\mathbb{R}\\ u=0 \text{ on } \partial U}} \int_U |\nabla u|^2\,dx \ \text{ subject to } \int_U u^2\,dx = 1.$$

Show that any minimizer is a solution of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } U\\ u = 0, & \text{on } \partial U \end{cases}$$

where $\lambda > 0$ is given by

$$\lambda = \int_U |\nabla u|^2\,dx.$$

[Hint: To show that $\lambda > 0$ and verify the formula for $\lambda$, multiply by $u$ on both sides of the PDE and integrate by parts.]

3. Show that the plane

$$u(x) = a \cdot x + b$$

solves the minimal surface equation on $U = \mathbb{R}^n$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

4. Show that for $n = 2$ the Scherk surface

$$u(x) = \log\left(\frac{\cos(x_1)}{\cos(x_2)}\right)$$

solves the minimal surface equation on the box $U = (-\frac{\pi}{2}, \frac{\pi}{2})^2$.