MATH 5588 – HOMEWORK 5 (DUE THURSDAY FEBRUARY 23)

1. Properties of the convolution.
   (a) Show that $(f * g)' = f' * g = f * g'$.
   (b) Show that $f * (g * h) = (f * g) * h$.
2. Compute the convolution of a Gaussian kernel $u(x) = e^{-x^2/2}$ with itself. [Hint: Use the Fourier transform]
3. Find the Fourier transform of the box function
$$u(x) = \begin{cases} 1, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1. \end{cases}$$
4. Use the Fourier transform to solve the damped heat equation $u_t - u_{xx} + u = 0$ with initial condition $u(x, 0) = f(x)$.
5. Write out Plancheral's identity for the box function from problem 3. What is
$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx?$$
6. **Nyquist-Shannon sampling theorem**: Let f be an integrable function whose Fourier transform $\widehat{f}$ vanishes outside of $(-\Omega, \Omega)$[1].
   (a) Show that when $\ell \geq \Omega$
$$\frac{1}{\sqrt{2\pi}}\widehat{f}(k) = \frac{1}{2\ell}\sum_{n=-\infty}^{\infty} f\left(\tfrac{n\pi}{\ell}\right) e^{-in\pi k/\ell} \quad \text{for } -\ell \leq k \leq \ell. \tag{1}$$
   [Hint: Write down the complex form of the Fouier series for $\widehat{f}$ on the interval $(-\ell, \ell)$. Simplify the coefficients with the inverse Fourier transform formula. You may assume that the Fourier series converges.]

   (b) Show that when $\ell \geq \Omega$
$$f(x) = \sum_{n=-\infty}^{\infty} f\left(\tfrac{n\pi}{\ell}\right) \operatorname{sinc}(\ell x - n\pi) \quad \text{for all } x \in \mathbb{R}, \tag{2}$$
   where $\operatorname{sinc}(x) := \sin(x)/x$ for $x \neq 0$, and $\operatorname{sinc}(0) := 1$. [Hint: Express f via the inverse Fourier transform formula and substitute the expression (1) for $\widehat{f}$. You may exchange the infinite summation and integral without justification.]

[1] This means $\widehat{f}(k) = 0$ for $|k| \geq \Omega$.

(c) Part (b) shows that f can be exactly reconstructed from evenly spaced samples $f\left(\frac{n\pi}{\ell}\right)$ for $n \in \mathbb{Z}$ provided $\ell \geq \Omega$. Find an expression for the *sampling frequency* f_s in terms of ℓ (use units of # samples per unit length). The *bandwidth* of f, denoted f_b, is the highest frequency (in units of cycles per unit length) present in the Fourier representation

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(k) e^{ikx}\, dk.$$

Since $\widehat{f}$ vanishes outside of $(-\Omega, \Omega)$, f_b is the frequency of the complex wave

$$e^{\pm i\Omega x} = \cos(\Omega x) \pm i \sin(\Omega x).$$

Find an expression for f_b, and show that $\ell \geq \Omega$ is equivalent to $f_s \geq 2f_b$. [This is the famous Nyquist-Shannon sampling theorem: A signal can be exactly reconstructed from evenly spaced samples provided the sampling frequency is at least twice the bandwidth of the signal. The reconstruction formula is given in (2), and is often called *sinc interpolation*.]