Math 5588 – Homework 6 (Due Thursday March 2)

This homework aims to examine the Fourier transform of generalized functions, or distributions, more rigorously. Recall the Fourier transform of a function $f(x)$ is

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx. \tag{1}$$

Since $|e^{-ikx}| = 1$ the integral on the right hand side converges absolutely only when

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty.$$

Functions f for which the integral above is finite are called *integrable*. The Fourier transform, as defined above, is only valid for integrable functions f. This leaves out many important functions, namely generalized functions, that we have been applying the Fourier transform to in class. For instance, the function $f(x) = 1$ is not integrable, so its Fourier transform does not appear to be defined (in class we found $\widehat{f}(k) = \sqrt{2\pi}\delta(k)$, which is a generalized function). The delta function $\delta(x)$ is not a function, so its Fourier transform is also not defined as above.

The aim of this homework is to define the Fourier transform for generalized functions and work out some related exercises. Recall a generalized function f is a continuous linear map from a space of test functions that we will call D to the real numbers. So for every test function $\varphi \in D$, $f(\varphi)$ is a real number, which we write as $\langle f, \varphi \rangle$ for convenience. We usually take D to contain all smooth compactly supported functions on $\mathbb{R}$, but other choices are important too, as we will see below. The delta function δ is a generalized function given by

$$\langle \delta, \varphi \rangle = \varphi(0).$$

Any locally integrable function[1] f defines a generalized function by the pairing

$$\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\, dx,$$

provided our test functions are compactly supported (i.e., vanishing outside of $[-R, R]$ for some $R > 0$).

1. Suppose f is a smooth integrable function. Show that for any smooth integrable test function $\varphi \in D$
$$\langle \widehat{f}, \varphi \rangle = \int_{-\infty}^{\infty} \widehat{f}(x)\varphi(x)\, dx = \langle f, \widehat{\varphi} \rangle.$$
[Note: This means the Fourier transform is *self-adjoint*.]

Based on problem 1, we *define* the Fourier transform of a generalized function f by the rule

$$\langle \widehat{f}, \varphi \rangle := \langle f, \widehat{\varphi} \rangle \quad \text{for all } \varphi \in D. \tag{2}$$

Note this rule only applies the Fourier transform (1) to the test function $\varphi \in D$, and not directly to the generalized function f, since this is not defined. We also write $\mathcal{F}(f) = \widehat{f}$ as usual.

[1] A function f is locally integrable if $\int_a^b |f(x)|\, dx < \infty$ for all real numbers $-\infty < a < b < \infty$.

2. Use the definition (2) to show that

$$\mathcal{F}(\delta(x)) = \frac{1}{\sqrt{2\pi}} \quad \text{and} \quad \mathcal{F}(1) = \sqrt{2\pi}\delta(k).$$

[Hint: Use the definition (2) to show that both sides act on test functions in the same way. At no point in time should you write anything like $\int_{-\infty}^{\infty} 1e^{-ikx}\,dx$ or $\int_{-\infty}^{\infty} \delta(x)e^{-ikx}\,dx$.]

3. What property should the class of test functions D have in order for (2) to be a sensible (or well-defined) definition? [Hint: The definition involves applying the generalized function f to $\widehat{\varphi}$.]

4. Show that a nonzero function and its Fourier transform cannot both be compactly supported. [Hint: A function is compactly supported if it is identically zero outside of a large enough interval $[-R, R]$. Show that only a finite number of terms in the function's Fourier **series** are nonzero. Therefore, the function must be a trigonometric polynomial. You can use the fact that a trigonometric polynomial cannot be identically zero on any interval (a, b) unless it is the zero polynomial (which follows from analyticity).]

Problem 4 shows that the class of smooth compactly supported test functions is too restrictive for defining the Fourier transform of generalized functions. The class commonly used is call the Schwartz space of functions, and it consists of functions that are smooth and rapidly decreasing as $x \to \pm\infty$ (but not necessarily compactly supported). This is a larger class of test functions.

5. The convolution

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy$$

is an important object when studying Fourier transforms. How would you define the convolution of a smooth compactly supported test function g with a *generalized function* f? [Hint: Assume at first that f is a smooth function and try to write something like

$$\langle f * g, \varphi\rangle = \int_{-\infty}^{\infty} (f * g)(x)\varphi(x)\,dx \cdots = \langle f, ?\rangle,$$

where you have to figure out what goes on the right hand side. Here, φ is a test function, and the right hand side should involve applying f to something involving g and φ. Then make the right hand side your definition of $f * g$ acting on φ.]