Math 5588 – Homework 9 (Due Thursday April 6)

1. Prove the Leibniz integral rule
$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)\,dt.$$
[Hint: For $a, b, x \in \mathbb{R}$ define
$$F(a,b,x) = \int_a^b f(x,t)\,dt,$$
and apply the multivariate chain rule
$$\frac{d}{dx}F(a(x),b(x),x) = \frac{\partial F}{\partial a}a'(x) + \frac{\partial F}{\partial b}b'(x) + \frac{\partial F}{\partial x}\cdot 1.$$
]

2. Solve the wave equation in three dimensions $n = 3$ with initial data $u(x,0) = 0$ and $u_t(x,0) = x_2$ using Kirchoff's formula.

3. Let $u(x,t)$ be a solution of the damped wave equation
$$\begin{cases} u_{tt} + \gamma u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = f & \text{on } \mathbb{R}^n \times \{t = 0\} \\ u_t = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$
where $\gamma \geq 0$. Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$, and define the backwards wave cone
$$K(x_0,t_0) := \Big\{(x,t) \,:\, 0 \leq t \leq t_0 \ \text{ and } \ |x - x_0| \leq t_0 - t\Big\}.$$
Prove that if $f \equiv g \equiv 0$ in $B(x_0,t_0) \times \{t = 0\}$, then $u \equiv 0$ in the cone $K(x_0,t_0)$. [Hint: Mimic the proof from class, in particular use the same energy.]

4. Repeat problem 3 for the nonlinear wave equation
$$u_{tt} - \Delta u + u^3 = 0.$$
[Hint: You will need to modify the energy to account for the u^3 term.]

5. Sketch a triangulation of the following domains so that all triangles have side length at most $\frac{1}{2}$:
   (a) A unit square.
   (b) An isosceles triangle with vertices $(-\frac{1}{2},0)$,$(\frac{1}{2},0)$, and $(0,1)$.
   (c) The square $[-2,2]^2$ with the hole $[-1,1]^2$ removed.
   (d) The unit disk.
   (e) The annulus $1 \leq |x| \leq 2$.

6. For a given vertex $v \in \mathbb{R}^2$ of a triangulation, the corresponding *vertex polygon* is the union of all triangles for which v is a vertex. Describe the vertex polygons for a triangulation that uses regular equilateral triangles.