MATH 5588 – HOMEWORK 9 SOLUTIONS

1. Prove the Leibniz integral rule

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)\,dt.$$

[Hint: For $a, b, x \in \mathbb{R}$ define

$$F(a,b,x) = \int_a^b f(x,t)\,dt,$$

and apply the multivariate chain rule

$$\frac{d}{dx}F(a(x),b(x),x) = \frac{\partial F}{\partial a}a'(x) + \frac{\partial F}{\partial b}b'(x) + \frac{\partial F}{\partial x}\cdot 1.$$

]

Solution. We have

$$\frac{\partial F}{\partial x}(a,b,x) = \int_a^b \frac{\partial f}{\partial x}(x,t)\,dt, \quad \frac{\partial F}{\partial a}(a,b,x) = -f(x,a) \ \text{ and } \ \frac{\partial F}{\partial b}(a,b,x) = f(x,b).$$

Applying the multivariate chain rule we have

$$\begin{aligned}\frac{d}{dx}F(a(x),b(x),x) &= \frac{\partial F}{\partial a}(a(x),b(x),x)a'(x) + \frac{\partial F}{\partial b}(a(x),b(x),x)b'(x) + \frac{\partial F}{\partial x}(a(x),b(x),x)\\ &= -f(x,a(x))a'(x) + f(x,b(x))b'(x) + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,t)\,dt.\end{aligned}$$

This completes the proof. □

2. Solve the wave equation in three dimensions $n = 3$ with initial data $u(x,0) = 0$ and $u_t(x,0) = x_2$ using Kirchoff's formula.

Solution. By Kirchoff's formula ($c = 1$)

$$u(x,t) = \frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} t\,y_2\,dS(y) = \frac{t}{|\partial B(x,t)|}\int_{\partial B(x,t)} y_2\,dS(y).$$

Since the function $v(y) = y_2$ is harmonic (all second derivatives vanish), the mean value formula gives

$$\frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} y_2\,dS(y) = \frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} v(y)\,dS(y) = v(x) = x_2.$$

Therefore $u(x) = x_2 t$. By the way, the solution is unchanged if $c \neq 1$. □

3. Let $u(x,t)$ be a solution of the damped wave equation

$$\begin{cases} u_{tt} + \gamma u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = f & \text{on } \mathbb{R}^n \times \{t=0\} \\ u_t = g & \text{on } \mathbb{R}^n \times \{t=0\}, \end{cases}$$

where $\gamma \geq 0$. Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$, and define the backwards wave cone

$$K(x_0,t_0) := \Big\{(x,t) \,:\, 0 \leq t \leq t_0 \;\text{ and }\; |x - x_0| \leq t_0 - t\Big\}.$$

Prove that if $f \equiv g \equiv 0$ in $B(x_0,t_0) \times \{t=0\}$, then $u \equiv 0$ in the cone $K(x_0,t_0)$. [Hint: Mimic the proof from class, in particular use the same energy.]

Solution. We mimic the proof from class. Define the energy

$$e(t) = \frac{1}{2}\int_{B(x,t_0-t)} u_t(x,t)^2 + |\nabla u(x,t)|^2\, dx.$$

Then as in the notes and class

$$\begin{aligned} \frac{de}{dt} &= -\frac{1}{2}\int_{\partial B(x,t_0-t)} u_t^2 + |\nabla u|^2\, dS + \int_{B(x,t_0-t)} u_t u_{tt} + \nabla u \cdot \nabla u_t\, dx \\ &= \int_{\partial B(x,t_0-t)} \frac{\partial u}{\partial \nu} u_t - \frac{1}{2}(u_t^2 + |\nabla u|^2)\, dS + \int_{B(x,t_0-t)} u_t u_{tt} - u_t \Delta u\, dx \\ &= \int_{\partial B(x,t_0-t)} \frac{\partial u}{\partial \nu} u_t - \frac{1}{2}(u_t^2 + |\nabla u|^2)\, dS + \int_{B(x,t_0-t)} u_t(u_{tt} - \Delta u)\, dx \\ &= \int_{\partial B(x,t_0-t)} \frac{\partial u}{\partial \nu} u_t - \frac{1}{2}(u_t^2 + |\nabla u|^2)\, dS - \gamma\int_{B(x,t_0-t)} u_t^2\, dx. \end{aligned}$$

Since $\gamma \geq 0$, the second term is non-positive (≤ 0), and by the same argument we made in class, the first term is also less than or equal to zero. Hence $de/dt \leq 0$, and the proof proceeds in the same way as in the notes from here. □

4. Repeat problem 3 for the nonlinear wave equation

$$u_{tt} - \Delta u + u^3 = 0.$$

[Hint: You will need to modify the energy to account for the u^3 term.]

Solution. The proof is similar to the previous problem, except here we use the energy

$$e(t) = \int_{B(x,t_0-t)} \frac{1}{2}u_t(x,t)^2 + \frac{1}{2}|\nabla u(x,t)|^2 + \frac{1}{4}u(x,t)^4\, dx.$$

Then we have

$$
\begin{aligned}
\frac{de}{dt} &= -\int_{\partial B(x,t_0-t)} \frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{4}u^4\,dS + \int_{B(x,t_0-t)} u_t u_{tt} + \nabla u\cdot\nabla u_t + u^3 u_t\,dx \\
&= \int_{\partial B(x,t_0-t)} \frac{\partial u}{\partial\nu}u_t - \frac{1}{2}u_t^2 - \frac{1}{2}|\nabla u|^2 - \frac{1}{4}u^4\,dS + \int_{B(x,t_0-t)} u_t u_{tt} - u_t\Delta u + u^3 u_t\,dx \\
&= \int_{\partial B(x,t_0-t)} \frac{\partial u}{\partial\nu}u_t - \frac{1}{2}u_t^2 - \frac{1}{2}|\nabla u|^2 - \frac{1}{4}u^4\,dS + \int_{B(x,t_0-t)} u_t(u_{tt} - \Delta u + u^3)\,dx \\
&= \int_{\partial B(x,t_0-t)} \frac{\partial u}{\partial\nu}u_t - \frac{1}{2}u_t^2 - \frac{1}{2}|\nabla u|^2 - \frac{1}{4}u^4\,dS.
\end{aligned}
$$

As in the notes

$$\frac{\partial u}{\partial\nu}u_t \le |\nabla u||u_t| \le \frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2.$$

Therefore

$$\frac{de}{dt} \le -\frac{1}{4}\int_{\partial B(x,t_0-t)} u^4\,dS \le 0,$$

and the proof proceeds in the same way as in the notes. □

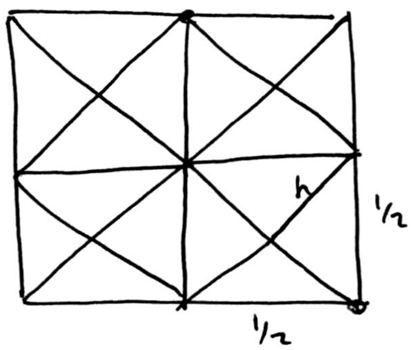
5. Sketch a triangulation of the following domains so that all triangles have side length at most $\frac{1}{2}$:
   (a) A unit square.
   (b) An isosceles triangle with vertices $(-\frac{1}{2},0)$,$(\frac{1}{2},0)$, and $(0,1)$.
   (c) The square $[-2,2]^2$ with the hole $[-1,1]^2$ removed.
   (d) The unit disk.
   (e) The annulus $1 \le |x| \le 2$.

Solution. See next page. □

6. For a given vertex $v \in \mathbb{R}^2$ of a triangulation, the corresponding *vertex polygon* is the union of all triangles for which v is a vertex. Describe the vertex polygons for a triangulation that uses regular equilateral triangles.

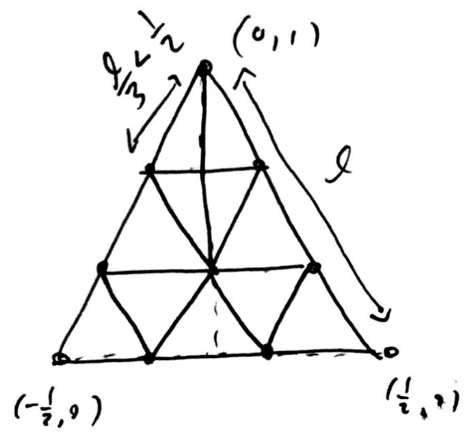
Solution. Equilateral triangles have equal angles of $\pi/3$. Each vertix is thus connected to exactly 6 triangles and the vertex polygons are regular hexagons (that is, the hexagon with equal interior angles). □

a)



$h^2 + h^2 = \frac{1}{4}$, $h = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}} < \frac{1}{2}$.

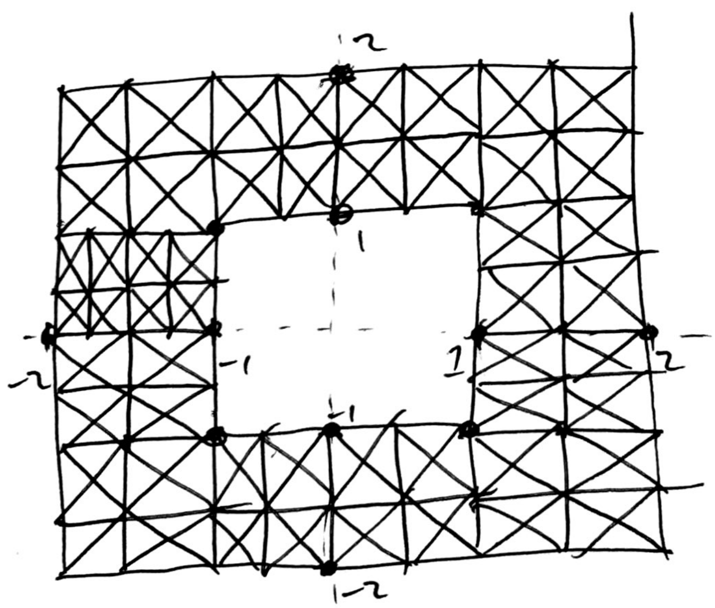
b)



$1^2 + (\frac{1}{2})^2 = \ell^2$

$\ell^2 = \frac{5}{4}$, $\ell = \frac{\sqrt{5}}{2}$

c)



Not to scale!

e) Similar...

d)