Math 5588 Midterm I

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Name:________________________________

Instructions:

1. I recommend looking over the problems first and starting with those you feel most comfortable with.

2. Unless otherwise noted, be sure to include explanations to justify each step in your arguments and computations. For example, make sure to check that the hypotheses of any theorems you use are satisfied.

3. All work should be done in the space provided in this exam booklet. Cross out any work you do not wish to be considered. Additional white paper is available if needed.

4. Books, notes, calculators, cell phones, pagers, or other similar devices are not allowed during the exam. Please turn off cell phones for the duration of the exam. You may use the formula sheet attached to this exam.

5. If you complete the exam within the last 15 minutes, please remain in your seat until the examination period is over.

6. In the event that it is necessary to leave the room during the exam (e.g., fire alarm), this exam and all your work must remain in the room, face down on your desk.

Problem	Score
1	/10
2	/10
3	/10
4	/10
5	/10
Total:	/50

1. Determine whether each statement below is true or false, and give a brief explanation why.

[10 points]

(a) [3 points] Solutions of the minimal surface equation are always unique.

(b) [3 points] The Euler-Lagrange equation for

$$I(u) = \int_0^1 u(x)u'(x)\,dx$$

is satisfied by every function $u : [0,1] \to \mathbb{R}$.

(c) [4 points] Every solution of the Euler-Lagrange equation for a functional of the form

$$I(u) = \int_0^1 L(u(x))\,dx$$

satisfies $L(u(x)) = C$ for some constant C.

2. [10 points] Find and solve the Euler-Lagrange equation for the functional

$$I(u) = \int_0^1 \log(u(x)u'(x))\,dx$$

subject to $u(1) = 1$ and $u(2) = \sqrt{2}$.

3. [10 points] Find the Euler-Lagrange equation for the constrained problem

$$\min_u I(u) = \int_U |\nabla u(x)|\, dx \quad \text{subject to} \quad J(u) = \int_U (f(x) - u(x))^2\, dx = 1,$$

where $C > 0$ is a constant and f is a given function. Here $U \subset \mathbb{R}^n$ is open and bounded and $u : U \to \mathbb{R}$.

4. [10 points] Consider the functional

$$I(u) = \int_0^1 \sqrt{u'(x)^2 + \varepsilon^2}\, dx,$$

where $\varepsilon > 0$. Find the function u that minimizes $I(u)$ subject to $u(0) = 0$ and $u(1) = 1$.

5. [10 points] Consider the functional

$$I(u) = \int_0^1 L(x, u(x), u'(x), u''(x))\, dx$$

where $L = L(x, z, p, q)$ (i.e., $z = u(x), p = u'(x)$, and $q = u''(x)$). We denote the partial derivatives of L by L_x, L_z, L_p, and L_q. Derive the Euler-Lagrange equation for I.

Scratch paper

Scratch paper

Scratch paper

Formula Sheet

$$L(u(x), u'(x)) - u'(x) L_p(u(x), u'(x)) = \text{Constant}$$

$$L_z(x, u(x), u'(x)) - \frac{d}{dx} L_p(x, u(x), u'(x)) = 0.$$

$$\nabla I(u) = L_z(x, u, \nabla u) - \operatorname{div}\left(\nabla_p L(x, u, \nabla u)\right) = 0$$

$$\int_U u_{x_i}\, dx = \int_{\partial U} u \nu_i \, dS.$$

$$\int_U u \Delta v \, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} \, dS - \int_U \nabla u \cdot \nabla v \, dx$$

$$\int_U u \Delta v - v \Delta u \, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \, dS$$

$$\int_U \Delta v \, dx = \int_{\partial U} \frac{\partial v}{\partial \nu} \, dS$$

$$\int_U u \operatorname{div}(F) \, dx = \int_{\partial U} u \, F \cdot \nu \, dS - \int_U \nabla u \cdot F \, dx.$$