## Math 5588 Midterm II Information

- The midterm will take place on Thursday, March 30, during class.
- The exam will cover lectures 8 through 15.
- The exam is closed book. No textbooks, notes, or calculators are allowed.
- The formula sheet at the end of these sample problems will be provided in the midterm.
- The exam will have 5 questions. The first 3 will be short, and the last 2 will be longer and slightly more involved. Below are a collection of sample midterm questions for you to practice.

## Sample questions

1. Give a brief answer to each question below.

   (a) Do you expect Fourier transform to be useful in general for studying Burger's equation
   $$u_t + u_x u = 0?$$
   Why or why not?

   (b) Would you expect the maximum principle to be useful for PDEs of order 3 or higher? Why or why not?

2. Suppose that $u(x)$ solves
   $$u + a \cdot \nabla u - \Delta u = f \quad \text{in } \mathbb{R}^n.$$
   Find $\widehat{u}(k)$.

3. Suppose that $u(x,t)$ solves
   $$u_t + \Delta^2 u = f.$$
   Find $\widehat{u}(k,t)$, were the Fourier transform is taken in $x$ only.

4. Recall the fundamental solution of the heat equation is given by
   $$\Phi^t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}.$$
   Use the Fourier transform to show that $\Phi^t * \Phi^s = \Phi^{t+s}$.

5. Suppose that $u(x,t)$ solves
   $$u_t + f * u = g,$$
   where $f(x)$ and $g(x)$ are given functions. Find $\widehat{u}(k,t)$, where the Fourier transform is taken in $x$ only.

6. For $t > 0$ let $u_m(x)$ be the solution of

$$\left(I - \tfrac{t}{m}\Delta\right)^m u = f \quad \text{in } \mathbb{R}^n.$$

Here, $I$ is the identity operator $Iu = u$.

(a) Show that

$$u_m(x) = S_m * f,$$

where

$$\widehat{S_m}(k) = \frac{1}{(2\pi)^{n/2}} \frac{1}{(1 + \frac{t}{m}|k|^2)^m}.$$

(b) Show that

$$\lim_{m\to\infty} \widehat{S_m}(k) = \frac{e^{-|k|^2 t}}{(2\pi)^{n/2}}.$$

[Hint: Recall that $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$.]

(c) Show that

$$\lim_{m\to\infty} u_m(x) = \Phi^t * f$$

where $\Phi^t$ is the fundamental solution of the heat equation

$$\Phi^t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}.$$

7. Suppose that

$$u - u\Delta u = 0 \quad \text{in } U$$

and $u(x) = 0$ for $x \in \partial U$, where $U \subseteq \mathbb{R}^n$ is open and bounded. Show that if $u \geq 0$ in $U$ then $u \equiv 0$. [Hint: Maximum principle.]

## Formula Sheet

$$\int_U u_{x_i}\,dx = \int_{\partial U} u\nu_i\,dS.$$

$$\int_U u\Delta v\,dx = \int_{\partial U} u\frac{\partial v}{\partial \nu}\,dS - \int_U \nabla u\cdot\nabla v\,dx$$

$$\int_U u\Delta v - v\Delta u\,dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\,dS$$

$$\int_U \Delta v\,dx = \int_{\partial U} \frac{\partial v}{\partial \nu}\,dS$$

$$\int_U u\,\mathrm{div}(F)\,dx = \int_{\partial U} u\,F\cdot\nu\,dS - \int_U \nabla u\cdot F\,dx.$$

$$\mathcal{F}(u) = \widehat{u}(k) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} u(x)e^{-ik\cdot x}\,dx.$$

$$\mathcal{F}^{-1}(\widehat{u}) = u(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \widehat{u}(k)e^{ik\cdot x}\,dk.$$

$$\mathcal{F}(u_{x_j}) = ik_j\widehat{u}(k).$$

$$\mathcal{F}(u*v) = (2\pi)^{n/2}\widehat{u}(k)\widehat{v}(k).$$

$$\int_{\mathbb{R}^n} |\widehat{u}(k)|^2\,dk = \int_{\mathbb{R}^n} |u(x)|^2\,dx.$$

$$\mathcal{F}(e^{-|x|^2/2}) = e^{-|k|^2/2}.$$