Math 8385 – Homework 1 (Due Oct 11)

1. Find and solve the Euler-Lagrange equation for the functional

$$I(u) = \int_0^{\log(2)} u(x)^2 + u'(x)^2 \, dx$$

subject to boundary conditions $u(0) = 0$ and $u(\log(2)) = 1$, where log is the natural logarithm.

2. Find the Euler-Lagrange equation for the functional

$$I(u) = \int_0^{\pi} u(x)^2 - (u'(x))^2 \, dx$$

subject to boundary conditions $u(0) = u(\pi) = 0$. Then find *all* solutions of the Euler-Lagrange equation (as a one-parameter family) and evaluate I on all solutions.

3. Let $1 \le p \le \infty$. The p-Laplacian is defined by

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$$

for $1 \le p < \infty$, and

$$\Delta_\infty u := \frac{1}{|\nabla u|^2} \sum_{i=1}^n \sum_{j=1}^n u_{x_i x_j} u_{x_i} u_{x_j}.$$

Notice that $\Delta_2 u = \Delta u$. A function u is called *p-harmonic* if $\Delta_p u = 0$.

(a) Show that the Euler-Lagrange equation for the functional

$$I(u) = \int_U \frac{1}{p} |\nabla u(x)|^p - u(x) f(x) \, dx$$

is the p-Laplace equation

$$-\Delta_p u = f \quad \text{in } U.$$

(b) Show that

$$\Delta_p u = |\nabla u|^{p-2} \left(\Delta u + (p-2)\Delta_\infty u\right).$$

(c) Show that

$$\Delta_\infty u = \lim_{p\to\infty} \frac{1}{p} |\nabla u|^{2-p} \Delta_p u.$$

4. Let $u, w \in C^2(\overline{U})$ where $U \subseteq \mathbb{R}^n$ is open and bounded. Show that for $1 \le p < \infty$

$$\int_U u \, \Delta_p w \, dx = \int_{\partial U} u \, |\nabla w|^{p-2} \frac{\partial w}{\partial \nu} \, dS - \int_U |\nabla w|^{p-2} \nabla u \cdot \nabla w \, dx.$$

[Hint: Use one of the integration by parts formulas in the appendix of the course notes.]

5. Find the Euler-Lagrange equation for the functional

$$I(u) = \int_U (\Delta u(x))^2 \, dx.$$

[Hint: Proceed as in the proof of the Euler-Lagrange equation from class. That is, let φ be smooth with compact support in U and compute

$$\frac{d}{dt}\Big|_{t=0} I(u + t\varphi) = 0.$$

Use integration by parts and the vanishing lemma to find the Euler-Lagrange equation.]

6. Consider the constrained problem

$$\min_{\substack{u:U\to\mathbb{R} \\ u=0 \text{ on } \partial U}} \int_U |\nabla u|^2 \, dx \quad \text{subject to} \quad \int_U u^2 \, dx = 1.$$

Show that any minimizer is a solution of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } U \\ \quad\; u = 0 & \text{on } \partial U, \end{cases} \tag{1}$$

where $\lambda > 0$ is given by

$$\lambda = \int_U |\nabla u|^2 \, dx.$$

[Hint: This does not require a Lagrange multiplier. Let u be a minimizer of the constrained problem, let $\varphi \in C_c^\infty(U)$, and consider the variation

$$t \mapsto w(x,t) := \frac{u(x) + t\varphi(x)}{\|u + t\varphi\|_{L^2(U)}}.$$

Since $\int_U w(x,t)^2 \, dx = 1$, we have $\frac{d}{dt}\Big|_{t=0} \int_U |\nabla w(x,t)|^2 \, dx = 0$. Complete the argument from here.]

7. Recall the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{in } U.$$

(a) Show that the plane

$$u(x) = a \cdot x + b$$

solves the minimal surface equation on $U = \mathbb{R}^n$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

(b) Show that for $n = 2$ the Scherk surface

$$u(x) = \log\left(\frac{\cos(x_1)}{\cos(x_2)} \right)$$

solves the minimal surface equation on the box $U = (-\frac{\pi}{2}, \frac{\pi}{2})^2$.