Math 8385 – Homework 1 Solutions

1. Find and solve the Euler-Lagrange equation for the functional

$$I(u) = \int_0^{\log(2)} u(x)^2 + u'(x)^2 \, dx$$

subject to boundary conditions $u(0) = 0$ and $u(\log(2)) = 1$, where log is the natural logarithm.

Solution. We use the form

$$L_z(x, u, u') - \frac{d}{dx} L_p(x, u, u') = 0$$

of the Euler-Lagrange equations. We have $L(x, z, p) = z^2 + p^2$ so $L_z = 2z$ and $L_p = 2p$. Therefore

$$0 = L_z(x, u, u') - \frac{d}{dx} L_p(x, u, u') = 2u(x) - \frac{d}{dx}(2u'(x)) = 2u(x) - 2u''(x).$$

Therefore the Euler-Lagrange equation is $u''(x) = u(x)$ and the general solution is

$$u(x) = Ae^x + Be^{-x}.$$

Using the boundary conditions yields

$$0 = u(0) = A + B \quad \text{and} \quad 1 = u(\log(2)) = 2A + \frac{1}{2}B.$$

Therefore $A = 2/3$ and $B = -2/3$, and

$$u(x) = \frac{2}{3}(e^x - e^{-x}) = \frac{4}{3}\sinh(x). \qquad \square$$

2. Find the Euler-Lagrange equation for the functional

$$I(u) = \int_0^{\pi} u(x)^2 - (u'(x))^2 \, dx$$

subject to boundary conditions $u(0) = u(\pi) = 0$. Then find *all* solutions of the Euler-Lagrange equation (as a one-parameter family) and evaluate I on all solutions.

Solution. As in problem 1 the Euler-Lagrange equation is $u''(x) = -u(x)$, and the general solution is

$$u(x) = A\sin(x) + B\cos(x).$$

The boundary conditions $u(0) = u(\pi) = 0$ imply that $B = 0$ but A can be arbitrary. So there is a family of solutions

$$u_A(x) = A\sin(x),$$

for every real number A. Since each u_A satisfies the Euler-Lagrange equation we have

$$\frac{d}{dA}\Big|_{A=1} I(A\sin(x)) = \frac{d}{dt}\Big|_{t=0} I(\sin(x) + t\sin(x)) = 0.$$

On the other hand $I(A\sin(x)) = A^2 I(\sin(x))$ and so

$$\frac{d}{dA}\Big|_{A=1} I(A\sin(x)) = 2I(\sin(x)).$$

Therefore $I(\sin(x)) = 0$ and

$$I(A\sin(x)) = A^2 I(\sin(x)) = 0 \quad \text{for all } A.$$

You can also directly compute $I(\sin(x))$ to show this. □

3. Let $1 \le p \le \infty$. The p-Laplacian is defined by

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

for $1 \le p < \infty$, and

$$\Delta_\infty u := \frac{1}{|\nabla u|^2}\sum_{i=1}^n\sum_{j=1}^n u_{x_i x_j} u_{x_i} u_{x_j}.$$

Notice that $\Delta_2 u = \Delta u$. A function u is called *p-harmonic* if $\Delta_p u = 0$.

(a) Show that the Euler-Lagrange equation for the functional

$$I(u) = \int_U \frac{1}{p}|\nabla u(x)|^p - u(x)f(x)\,dx$$

is the p-Laplace equation

$$-\Delta_p u = f \quad \text{in } U.$$

Solution. Here

$$L(x, z, q) = \frac{1}{p}|q|^p - zf(x).$$

[It was a bad choice of notation to use p for the power in the p-Laplace equation when we also use p for ∇u]. Therefore $L_z(x, z, q) = -f(x)$ and as in problem 3

$$\nabla_q L(x, z, q) = |q|^{p-1}\frac{q}{|q|} = |q|^{p-2}q.$$

Therefore the Euler-Lagrange equation is

$$-f(x) - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0.$$

Using the definition of the p-Laplace equation we have

$$-\Delta_p u = f.$$

□

(b) Show that

$$\Delta_p u = |\nabla u|^{p-2} \left(\Delta u + (p-2) \Delta_\infty u \right).$$

Solution. By the product rule we have

$$\begin{aligned}
\Delta_p u &= |\nabla u|^{p-2} \operatorname{div}(\nabla u) + \nabla(|\nabla u|^{p-2}) \cdot \nabla u \\
&= |\nabla u|^{p-2} \Delta u + \nabla(|\nabla u|^{p-2}) \cdot \nabla u.
\end{aligned}$$

Hence, all we need to show is that

$$\nabla(|\nabla u|^{p-2}) \cdot \nabla u = |\nabla u|^{p-2} (p-2) \Delta_\infty u.$$

To compute this, note that

$$|\nabla u|^{p-2} = (u_{x_1}^2 + \cdots + u_{x_n}^2)^{\frac{p-2}{2}}.$$

Therefore

$$\begin{aligned}
\frac{\partial}{\partial x_i} |\nabla u|^{p-2} &= \frac{\partial}{\partial x_i} (u_{x_1}^2 + \cdots + u_{x_n}^2)^{\frac{p-2}{2}} \\
&= \frac{p-2}{2} (u_{x_1}^2 + \cdots + u_{x_n}^2)^{\frac{p-2}{2}-1} \frac{\partial}{\partial x_i} (u_{x_1}^2 + \cdots + u_{x_n}^2) \\
&= \frac{p-2}{2} (u_{x_1}^2 + \cdots + u_{x_n}^2)^{\frac{p-4}{2}} (2u_{x_1} u_{x_1 x_i} + 2u_{x_2} u_{x_2 x_i} + \cdots + 2u_{x_n} u_{x_n x_i}) \\
&= (p-2) |\nabla u|^{p-4} (u_{x_1} u_{x_1 x_i} + u_{x_2} u_{x_2 x_i} + \cdots + u_{x_n} u_{x_n x_i}) \\
&= (p-2) |\nabla u|^{p-4} \sum_{j=1}^n u_{x_j} u_{x_j x_i}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\nabla(|\nabla u|^{p-2}) \cdot \nabla u &= \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} |\nabla u|^{p-2} \right) u_{x_i} \\
&= \sum_{i=1}^n (p-2) |\nabla u|^{p-4} \sum_{j=1}^n u_{x_j} u_{x_j x_i} u_{x_i} \\
&= (p-2) |\nabla u|^{p-4} \sum_{i=1}^n \sum_{j=1}^n u_{x_i x_j} u_{x_i} u_{x_j} \\
&= (p-2) |\nabla u|^{p-2} \Delta_\infty u.
\end{aligned}$$

This completes the proof. □

(c) Show that

$$\Delta_\infty u = \lim_{p \to \infty} \frac{1}{p} |\nabla u|^{2-p} \Delta_p u.$$

Solution. By part (a) we have

$$\frac{1}{p}|\nabla u|^{2-p}\Delta_p u = \frac{1}{p}\Delta u + \frac{p-2}{p}\Delta_\infty u.$$

Sendig $p \to \infty$ we have

$$\lim_{p\to\infty}\frac{1}{p}|\nabla u|^{2-p}\Delta_p u = \lim_{p\to\infty}\frac{1}{p}\Delta u + \lim_{p\to\infty}\frac{p-2}{p}\Delta_\infty u = \Delta_\infty u. \qquad \square$$

4. Let $u, w \in C^2(\overline{U})$ where $U \subseteq \mathbb{R}^n$ is open and bounded. Show that for $1 \le p < \infty$

$$\int_U u\,\Delta_p w\,dx = \int_{\partial U} u\,|\nabla w|^{p-2}\frac{\partial w}{\partial \nu}\,dS - \int_U |\nabla w|^{p-2}\nabla u \cdot \nabla w\,dx.$$

[Hint: Use one of the integration by parts formulas in the appendix of the course notes.]

Solution. Recall that

$$\Delta_p u = \text{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

By the divergence theorem, or integration by parts, we have

$$\begin{aligned}\int_U u\,\Delta_p w\,dx &= \int_U u\,\text{div}\left(|\nabla w|^{p-2}\nabla w\right)\,dx \\ &= \int_{\partial U} u|\nabla w|^{p-2}\nabla w \cdot \nu\,dS - \int_U \nabla u \cdot |\nabla w|^{p-2}\nabla w\,dx \\ &= \int_{\partial U} u|\nabla w|^{p-2}\frac{\partial w}{\partial \nu}\,dS - \int_U |\nabla w|^{p-2}\nabla u \cdot \nabla w\,dx.\end{aligned}$$

$\square$

5. Find the Euler-Lagrange equation for the functional

$$I(u) = \int_U (\Delta u(x))^2\,dx.$$

[Hint: Proceed as in the proof of the Euler-Lagrange equation from class. That is, let φ be smooth with compact support in U and compute

$$\left.\frac{d}{dt}\right|_{t=0} I(u + t\varphi) = 0.$$

Use integration by parts and the vanishing lemma to find the Euler-Lagrange equation.]

Solution. Suppose u is a min or max of I subject to any boundary conditions (our compactly supported test function ignores the boundary conditions). Let φ be smooth with compact support and compute

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} I(u + t\varphi) \\ &= \int_U \frac{d}{dt}\Big|_{t=0} (\Delta u + t\Delta\varphi)^2 \, dx \\ &= \int_U 2\Delta u \Delta\varphi \, dx. \end{aligned}$$

Therefore

$$\int_U \Delta u \Delta \varphi \, dx = 0$$

for all test functions φ with compact support in U. Using Green's second identity (or integration by parts) we have

$$0 = \int_U \Delta u \Delta \varphi \, dx = \int_U (\Delta\Delta u)\varphi \, dx$$

due to the fact that φ and $\nabla\varphi$ vanish on the boundary ∂U. By the vanishing lemma

$$\Delta^2 u = 0 \quad \text{in } U$$

where $\Delta^2 u = \Delta\Delta u$. This is called the bi-harmonic equation, and is the Euler-Lagrange equation for the functional in question. Written out the operator is

$$\Delta^2 u = \sum_{i=1}^n \sum_{j=1}^n u_{x_i x_i x_j x_j} = 0$$

which is a fourth order partial differential equation. □

6. Consider the constrained problem

$$\min_{\substack{u:U\to\mathbb{R} \\ u=0 \text{ on } \partial U}} \int_U |\nabla u|^2 \, dx \quad \text{subject to} \quad \int_U u^2 \, dx = 1.$$

Show that any minimizer is a solution of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } U \\ \quad u = 0 & \text{on } \partial U, \end{cases} \tag{1}$$

where $\lambda > 0$ is given by

$$\lambda = \int_U |\nabla u|^2 \, dx.$$

[Hint: This does not require a Lagrange multiplier. Let u be a minimizer of the constrained problem, let $\varphi \in C_c^\infty(U)$, and consider the variation

$$t \mapsto w(x,t) := \frac{u(x) + t\varphi(x)}{\|u + t\varphi\|_{L^2(U)}}.$$

Since $\int_U w(x,t)^2\,dx = 1$, we have $\frac{d}{dt}\Big|_{t=0} \int_U |\nabla w(x,t)|^2\,dx = 0$. Complete the argument from here.]

Solution. We are minimizing $I(u) = \int_U |\nabla u|^2\,dx$ subject to $J(u) = \int_U u^2\,dx - 1 = 0$, and we have

$$\nabla I(u) = -2\Delta u \quad \text{and} \quad \nabla J(u) = 2u.$$

The Euler-Lagrange equations for the constrained problem are therefore

$$-2\Delta u + \overline{\lambda} 2u = 0$$

for some Lagrange multiplier $\overline{\lambda}$. Setting $\lambda = -\overline{\lambda}$ we have

$$-\Delta u = \lambda u.$$

Multiplying by u and integrating over U we have

$$-\int_U u\Delta u\,dx = \lambda \int_U u^2\,dx.$$

Since u satisfies $u = 0$ on ∂U and $\int_U u^2 = 1$ we can integrate by parts to obtain

$$\int_U |\nabla u|^2\,dx = -\int_U u\Delta u\,dx = \lambda \int_U u^2\,dx = \lambda.$$

This verifies that $\lambda > 0$ (since u cannot be constant), and $\lambda = \int_U |\nabla u|^2\,dx$. □

7. Recall the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{in } U.$$

(a) Show that the plane

$$u(x) = a \cdot x + b$$

solves the minimal surface equation on $U = \mathbb{R}^n$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

Solution. This is obvious. □

(b) Show that for $n = 2$ the Scherk surface

$$u(x) = \log\left(\frac{\cos(x_1)}{\cos(x_2)}\right)$$

solves the minimal surface equation on the box $U = (-\frac{\pi}{2}, \frac{\pi}{2})^2$.

Solution. Notice that

$$u(x) = \log(\cos(x_1)) - \log(\cos(x_2)) = f(x_1) - f(x_2)$$

where $f(x) = \log(\cos(x))$. Let's just look for a solution in the form

$$u(x) = f(x_1) - f(x_2),$$

and see what we get. Note that $u_{x_1x_2} = 0$, $u_{x_1} = f'(x_1)$, $u_{x_2} = -f'(x_2)$, $u_{x_1x_1} = f''(x_1)$, and $u_{x_2x_2} = f''(x_2)$. Plugging this into the minimal surface equation

$$(1 + u_{x_2}^2)u_{x_1x_1} - 2u_{x_1x_2}u_{x_1}u_{x_2} + (1 + u_{x_1}^2)u_{x_2x_2} = 0$$

we get

$$(1 + f'(x_2)^2)f''(x_1) - (1 + f'(x_1)^2)f''(x_2) = 0.$$

Separating variables we have

$$\frac{f''(x_1)}{1 + f'(x_1)^2} = \frac{f''(x_2)}{1 + f'(x_2)^2}.$$

Since the left side depends only on x_1 and the right side only on x_2, both sides must be equal to the same constant A, hence

$$\frac{f''(x)}{1 + f'(x)^2} = A.$$

Therefore

$$\arctan(f'(x)) = Ax + B,$$

and $f'(x) = \tan(Ax + B)$. Note we must restrict A and B so that

$$-\frac{\pi}{2} < Ax + B < \frac{\pi}{2}$$

for all $x \in (-\pi/2, \pi/2)$. We can assume without loss of generality that $A < 0$ (or else we can consider $-f$), so we need

$$-\frac{\pi}{2}A + B \leq \frac{\pi}{2}$$

and

$$\frac{\pi}{2}A + B \geq -\frac{\pi}{2}.$$

Together these imply that $-1 \leq A \leq 0$ and $|B| \leq \frac{\pi}{2}(A + 1)$ (so once we choose A we have a obtain a range of choices for B.)
Integrating $f'(x) = \tan(Ax + B)$ we have

$$f(x) = -\frac{1}{A}\log(\cos(Ax + B)).$$

So in general we have the minimal surface

$$u(x) = -\frac{1}{A}\log(\cos(Ax_1 + B)) + \frac{1}{A}\log(\cos(Ax_2 + B)),$$

subject to $-1 \leq A \leq 0$ and $B| \leq \frac{\pi}{2}(A + 1)$. Note we can write

$$u(x) = -\frac{1}{A}\log\left(\frac{\cos(Ax_1 + B)}{\cos(Ax_2 + B)}\right).$$

The Scherk surface is obtained by selecting $A = -1$ and $B = 0$. □