Note: Please choose a paper for your term end project/presentation by Friday Oct 5.

MATH 8590 – HOMEWORK 2 (DUE FRIDAY OCT 5)

Please hand in your solution to 1 problem from those below.
Let $U \subset \mathbb{R}^n$ be open.

1. (a) Let $u, v \in \mathrm{USC}(\overline{U})$. Suppose that $w := u$ and $w := v$ are viscosity solutions of

$$H(D^2w, Dw, w, x) \leq 0 \text{ in U.} \tag{1}$$

   Show that $w(x) := \max\{u(x), v(x)\}$ is a viscosity solution of (1) (i.e., the pointwise maximum of two subsolutions is again a subsolution).

   (b) Let $u, v \in \mathrm{LSC}(\overline{U})$. Suppose that $w := u$ and $w := v$ are viscosity solutions of

$$H(D^2w, Dw, w, x) \geq 0 \text{ in U.} \tag{2}$$

   Show that $w(x) := \min\{u(x), v(x)\}$ is a viscosity solution of (2).

2. For each $k \in \mathbb{N}$, let $u_k \in C(U)$ be a viscosity solution of

$$H(D^2u_k, Du_k, u_k, x) = 0 \text{ in } U.$$

   Suppose that $u_k \to u$ locally uniformly on U (this means $u_k \to u$ uniformly on every $V \subset\subset U$). Show that u is a viscosity solution of

$$H(D^2u, Du, u, x) = 0 \text{ in } U.$$

   Thus, viscosity solutions are stable under uniform convergence. (We will see shortly that viscosity solutions are stable under even weaker types of convergence.)

3. Suppose that $H = H(p, x)$ is continuous and $p \mapsto H(p, x)$ is *convex* for any fixed x. Let $u \in C^{0,1}_{loc}(U)$ satisfy

$$\lambda u(x) + H(Du(x), x) \leq 0 \quad \text{for a.e. } x \in U,$$

   where $\lambda \geq 0$. Show that u is a viscosity solution of

$$\lambda u + H(Du, x) \leq 0 \text{ in } U.$$

   Give an example to show that the same result does not hold for supersolutions. [Hint: Mollify u: $u_\varepsilon := \eta_\varepsilon * u$. For $V \subset\subset U$, use Jensen's inequality to show that

$$\lambda u_\varepsilon(x) + H(Du_\varepsilon(x), x) \leq h_\varepsilon(x) \text{ for all } x \in V$$

   and $\varepsilon > 0$ sufficiently small, where $h_\varepsilon \to 0$ uniformly on V. Then apply an argument similar to problem 2.]

4. Let $1 < p < \infty$ and define

$$|x|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}.$$

Assume $U \subset \mathbb{R}^n$ is open, bounded, and path connected with Lipschitz boundary ∂U, and let $f : \overline{U} \to \mathbb{R}$ be continuous and positive. Show that there exists a unique viscosity solution $u \in C(\overline{U})$ of the p-eikonal equation

$$\text{(P)} \quad \begin{cases} |Du|_p = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

[Hint: Construct u as the value function

$$u(x) = \inf\{T(x, y) \,:\, y \in \partial U\},$$

where

$$T(x, y) = \inf \left\{ \int_0^1 f(\mathbf{w}(t)) |\mathbf{w}'(t)|_q \, dt \,:\, \mathbf{w} \in C^1([0,1]; \overline{U}),\, \mathbf{w}(0) = x,\, \mathbf{w}(1) = y \right\},$$

and q is the Hölder conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Don't worry about exactly computing the form of H. Instead show that any solution of $H = 0$ is also a solution of (P).]