# Math 8590 – Homework 3 (Due Friday Oct 19)

Please hand in your solution to 1 problem from those below.

1. Complete the proof of Theorem 5.2 in the notes. In particular, let $u_\varepsilon \in C^2(U) \cap C(\overline{U})$ be a classical solution of the viscous Hamilton-Jacobi equation

$$(\mathrm{H}_\varepsilon) \quad \begin{cases} u_\varepsilon + H(Du_\varepsilon, x) - \varepsilon \Delta u_\varepsilon = 0 & \text{in } U \\ u_\varepsilon = 0 & \text{on } \partial U, \end{cases}$$

and let $u \in C^{0,1}(\overline{U})$ be the unique viscosity solution of

$$(\mathrm{H}) \quad \begin{cases} u + H(Du, x) = 0 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Assume that $H$ and $U$ satisfy all of the hypotheses stated at the beginning of Section 5 in the notes. Show that there exists $C > 0$ such that

$$u_\varepsilon - u \leq C\sqrt{\varepsilon}.$$

[Hint: Define the auxiliary function

$$\Phi(x, y) = u_\varepsilon(x) - u(y) - \frac{\alpha}{2}|x - y|^2,$$

and proceed as in the proof of Theorem 5.2. You will need to use the exterior sphere condition and the barrier function method from the proof of Theorem 5.1 to handle the case when $y_\alpha \in \partial U$. For the exterior sphere condition, you can assume that the same radius $r > 0$ works for all boundary points.]

2. (a) Let $u \in C(\overline{U})$ be a viscosity solution of

$$H(Du, u, x) = 0 \quad \text{in } U.$$

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with $\Psi' > 0$. Show that $v := \Psi \circ u$ is a viscosity solution of

$$H((\Phi' \circ v)Dv, \Phi \circ v, x) = 0 \ \text{ in } U,$$

where $\Phi := \Psi^{-1}$.

(b) Let $u \in C(\overline{U})$ be a viscosity solution of

$$H(Du) = f \ \text{ in } U,$$

and suppose that $H$ is positively 1-homogeneous. Define the Kružkov Transform of $u$ by $v := -e^{-u}$. Use part (a) to show that $v$ is a viscosity solution of

$$fv + H(Dv) = 0 \ \text{ in } U. \tag{1}$$

[Remark: The Kružkov Transform is a standard technique for introducing a zeroth order term. When $f > 0$, this term has the correct sign for a comparison principle to hold for (1). This also shows that we do not lose much in the way of generality by studying equations with zeroth order terms.]

3. Consider the Hamilton-Jacobi equation

$$u + H(Du, x) = 0 \text{ in } \mathbb{R}^n.$$

What (non-trivial) conditions can you place on $H$ to guarantee the existence of a bounded viscosity solution $u \in C(\mathbb{R}^n)$? State your conditions and give the proof. [Hint: Use the Perron method.]