Math 8590 – Homework 3 Solutions

Please hand in your solution to 1 problem from those below.

1. Complete the proof of Theorem 5.2 in the notes. In particular, let $u_\varepsilon \in C^2(U) \cap C(\overline{U})$ be a classical solution of the viscous Hamilton-Jacobi equation

$$(\mathrm{H}_\varepsilon) \quad \begin{cases} u_\varepsilon + H(Du_\varepsilon, x) - \varepsilon \Delta u_\varepsilon = 0 & \text{in } U \\ u_\varepsilon = 0 & \text{on } \partial U, \end{cases}$$

and let $u \in C^{0,1}(\overline{U})$ be the unique viscosity solution of

$$(\mathrm{H}) \quad \begin{cases} u + H(Du, x) = 0 & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Assume that H and U satisfy all of the hypotheses stated at the beginning of Section 5 in the notes. Show that there exists $C > 0$ such that

$$u_\varepsilon - u \le C\sqrt{\varepsilon}.$$

[Hint: Define the auxiliary function

$$\Phi(x, y) = u_\varepsilon(x) - u(y) - \frac{\alpha}{2}|x - y|^2,$$

and proceed as in the proof of Theorem 5.2. You will need to use the exterior sphere condition and the barrier function method from the proof of Theorem 5.1 to handle the case when $y_\alpha \in \partial U$. For the exterior sphere condition, you can assume that the same radius $r > 0$ works for all boundary points.]

Solution. Define the auxiliary function

$$\Phi(x, y) = u_\varepsilon(x) - u(y) - \frac{\alpha}{2}|x - y|^2.$$

Let $(x_\alpha, y_\alpha) \in \overline{U} \times \overline{U}$ such that

$$\Phi(x_\alpha, y_\alpha) = \max_{\overline{U}\times\overline{U}} \Phi.$$

Since $\Phi(x_\alpha, y_\alpha) \ge \Phi(x_\alpha, x_\alpha)$ we have

$$\frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \le u(x_\alpha) - u(y_\alpha) \le C|x_\alpha - y_\alpha|.$$

Therefore

$$|x_\alpha - y_\alpha| \le \frac{C}{\sqrt{\alpha}}.$$

Let us set $\alpha = \frac{1}{\sqrt{\varepsilon}}$. Then we have

$$|x_\alpha - y_\alpha| \le C\sqrt{\varepsilon}.$$

As in the proof of Theorem 5.2, we just need to show that

$$u_\varepsilon(x_\alpha) - u(y_\alpha) \le C\sqrt{\varepsilon}.$$

We have 3 cases to consider

1. Suppose $x_\alpha \in \partial U$. Then $u_\varepsilon(x_\alpha) = 0$ and $u(y_\alpha) \ge 0$ so we have

$$u_\varepsilon(x_\alpha) - u(y_\alpha) \le 0.$$

2. Suppose that $y_\alpha \in \partial U$. Then $u(y_\alpha) = 0$ and we have

$$u_\varepsilon(x_\alpha) - u(y_\alpha) = u_\varepsilon(x_\alpha).$$

We need to use the barrier function technique from the proof of Theorem 5.1 to bound the right hand side. By coercivity of H we can select $C' > 0$ and $\delta > 0$ so that

$$H(C'p, x) \ge \delta \text{ for all } x \in U \text{ and } |p| = 1.$$

By the exterior sphere condition there exists $r > 0$ and $x_0 \in \mathbb{R}^n \setminus \overline{U}$ such that $|x_0 - y_\alpha| = r$ and

$$\psi(x) := C'(|x - x_0| - r) \ge 0 \quad \text{for all } x \in \overline{U}.$$

We note that

$$|D\psi(x)| = C' \quad \text{and} \quad \Delta\psi(x) = \frac{C'(n-1)}{|x - x_0|} \le \frac{C'(n-1)}{r}$$

for all $x \in U$. Since the same r works for all boundary points, the quantity $C'(n-1)/r$ depends only on U and n. It follows that

$$\psi(x) + H(D\psi(x), x) - \varepsilon\Delta\psi(x) \ge \delta - C\varepsilon$$

for all $x \in U$. Hence there exists $\varepsilon' > 0$, depending only on H, U and n, such that for $0 < \varepsilon < \varepsilon'$ we have

$$\psi(x) + H(D\psi(x), x) - \varepsilon\Delta\psi(x) \ge 0 \quad \text{for all } x \in U,$$

and $\psi \ge 0$ on ∂U. Therefore we can use maximum principle arguments to show that $u_\varepsilon \le \psi$ on $\overline{U}$ for $\varepsilon < \varepsilon'$. Therefore

$$u_\varepsilon(x_\alpha) \le \psi(x_\alpha) = C(|x_\alpha - x_0| - r) \le C|x_\alpha - y_\alpha| = C\sqrt{\varepsilon}.$$

For $\varepsilon \ge \varepsilon'$ we have

$$u_\varepsilon(x_\alpha) \le \left(\sup_{\varepsilon > 0, x \in U} u_\varepsilon \right) \frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon'}} = C\sqrt{\varepsilon}.$$

3. Suppose that $x_\alpha, y_\alpha \in U$. By the viscosity supersolution property, we have

$$u(y_\alpha) + H(p_\alpha, y_\alpha) \ge 0,$$

where $p_\alpha = \alpha(x_\alpha - y_\alpha)$. Since $x \mapsto u_\varepsilon(x) - \frac{\alpha}{2}|x - y_\alpha|^2$ has a maximum at x_α we have $Du_\varepsilon(x_\alpha) = p_\alpha$ and $\Delta u_\varepsilon(x_\alpha) \leq \alpha n$. Therefore

$$u_\varepsilon(x_\alpha) + H(p_\alpha, x_\alpha) - \alpha n \varepsilon \leq 0.$$

Subtracting these inequalities we have

$$u_\varepsilon(x_\alpha) - u(y_\alpha) \leq H(p_\alpha, y_\alpha) - H(p_\alpha, x_\alpha) + C\sqrt{\varepsilon} \leq C\sqrt{\varepsilon},$$

due to the Lipschitz continuity of H. This completes the proof. □

2. (a) Let $u \in C(\overline{U})$ be a viscosity solution of

$$H(Du, u, x) = 0 \quad \text{in } U.$$

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with $\Psi' > 0$. Show that $v := \Psi \circ u$ is a viscosity solution of

$$H((\Phi' \circ v)Dv, \Phi \circ v, x) = 0 \ \text{ in } U,$$

where $\Phi := \Psi^{-1}$.

Solution. Let $x \in U$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $v - \varphi$ has a local maximum at x. We may assume that $v(x) = \varphi(x)$. Then there exists $r > 0$ such that $v(y) \leq \varphi(y)$ for $y \in B(x, r)$. Let $\Phi := \Psi^{-1}$. Since Φ is increasing we have $u(y) = \Phi(v(y)) \leq \Phi(\varphi(y)) := \psi(y)$ for all $y \in B(x, r)$, and $u(x) = \Phi(v(x)) = \Phi(\varphi(x)) = \psi(x)$. Therefore $u - \psi$ has a local maximum at x, and therefore

$$H(D\psi(x), u(x), x) \leq 0.$$

Since $D\psi(x) = \Phi'(v(x))D\varphi(x)$ we have

$$H(\Phi'(v(x))D\varphi(x), \Phi(v(x)), x) \leq 0.$$

This verifies the subsolution property. The supersolution property follows a similar argument. □

(b) Let $u \in C(\overline{U})$ be a viscosity solution of

$$H(Du) = f \ \text{ in } U,$$

and suppose that H is positively 1-homogeneous. Define the Kružkov Transform of u by $v := -e^{-u}$. Use part (a) to show that v is a viscosity solution of

$$fv + H(Dv) = 0 \ \text{ in } U. \tag{1}$$

[Remark: The Kružkov Transform is a standard technique for introducing a zeroth order term. When $f > 0$, this term has the correct sign for a comparison principle to hold for (1). This also shows that we do not lose much in the way of generality by studying equations with zeroth order terms.]

Solution. We are in the same setting as part (a), where $\Psi(x) = -e^{-x}$ and $\Phi(x) = -\log(-x)$. Therefore v is a viscosity solution of

$$H\left(-\frac{Dv}{v}\right) = f \quad \text{in } U.$$

Let $x \in U$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $v - \varphi$ has a local maximum at x. Then

$$H\left(-\frac{D\varphi(x)}{v(x)}\right) \leq f(x) \quad \text{in } U.$$

Since H is positively one homogeneous and $v(x) < 0$ we have

$$f(x) \geq H\left(-\frac{D\varphi(x)}{v(x)}\right) = -\frac{1}{v(x)}H(D\varphi(x)).$$

Therefore

$$f(x)v(x) + H(D\varphi(x)) \leq 0.$$

The supersolution property is verified similarly. □

3. Consider the Hamilton-Jacobi equation

$$u + H(Du, x) = 0 \quad \text{in } \mathbb{R}^n.$$

What (non-trivial) conditions can you place on H to guarantee the existence of a bounded viscosity solution $u \in C(\mathbb{R}^n)$? State your conditions and give the proof. [Hint: Use the Perron method.]

Solution. Perron's method requires a comparison principle, so we assume the usual conditions

$$H(p, x) - H(p, y) \leq \omega_1((1 + |p|)|x - y|)$$

and

$$H(p, x) - H(q, x) \leq \omega_2(|p - q|).$$

In order to construct the bounded super and subsolutions required in Perron's method, we also need to assume

$$K := \sup\{|H(0, x)| : x \in \mathbb{R}^n\} < \infty.$$

Let $w(x) = K$ and $\tilde{w}(x) = -K$. Then w is a viscosity supersolution and $\tilde{w}$ is a viscosity subsolution. Define

$$\mathcal{F} := \{v \in \text{USC}(\mathbb{R}^n) : v \text{ is a subsolution and } v \leq w\},$$

and

$$u(x) := \sup\{v(x) : v \in \mathcal{F}\}.$$

Since $\tilde{w} \in \mathcal{F}$, $\mathcal{F}$ is nonempty, and we have u^* is a viscosity subsolution of $u + H = 0$ on $\mathbb{R}^n$ (recall u^* is the upper semicontinuous envelope of u). Since $u \leq w$ we have $u^* \leq w$

due to the fact that w is smooth. Therefore $u^* \in \mathcal{F}$ and $u^* = u$. The other lemma from Perron's method says that u_* is a viscosity supersolution of $u + H = 0$. Define $u_\varepsilon := u - \varepsilon$. Then we can show that

$$u_\varepsilon + H(Du_\varepsilon, x) + \varepsilon \leq 0 \quad \text{in } \mathbb{R}^n$$

in the viscosity sense. Indeed, if $u_\varepsilon - \varphi$ has a local maximum at $x \in \mathbb{R}^n$, then $u - (\varphi + \varepsilon)$ has a local maximum at x and thus

$$u(x) + H(D\varphi(x), x) \leq 0.$$

Therefore

$$u_\varepsilon(x) + H(D\varphi(x), x) + \varepsilon \leq 0.$$

By the comparison theorem with strict sub solution (Theorem 6.1 in notes), we have $u_\varepsilon \leq u_*$. Sending $\varepsilon \to 0^+$ we have $u \leq u^*$. Since $u = u^* \geq u_*$, we have that $u = u^* = u_* \in C(\mathbb{R}^n)$ is a bounded viscosity solution of $u + H = 0$ in $\mathbb{R}^n$, and u satisfies $-K \leq u \leq K$. $\square$