Math 8590 – Homework 4 (Due Friday Nov 2)

Please hand in your solution to 1 problem from those below.

1. Implement both fast sweeping and fast marching in the programming language of your choice for solving the one dimensional eikonal equation

\[ |u'(x)| = f(x) \text{ for } x \in (0,1), \]

with boundary conditions \( u(0) = u_0 \) and \( u(1) = u_1 \). Experiment with different functions \( f \geq 0 \) and different boundary conditions. Are the boundary conditions always attained continuously? Which method is faster? Produce some plots of solutions to hand in, and turn in a working version of your code.

2. Suppose the numerical solutions \( u_h \) of our monotone scheme \( S_h = 0 \) are uniformly Lipschitz continuous, i.e., there exists \( C > 0 \) such that

\[ |u_h(x) - u_h(y)| \leq C|x - y| \text{ for all } x, y \in [0,1]^n_h \text{ and } h > 0. \]

This is a stronger form of stability. Prove the Barles-Souganidis convergence theorem from class without assuming strong uniqueness. You can assume that ordinary uniqueness holds, that is, there is at most one viscosity solution satisfying the boundary conditions in the usual sense. [Hint: Use the Arzelà-Ascoli Theorem to extract a subsequence \( u_{h_k} \) converging uniformly to a continuous function \( u \in C([0,1]^n) \). Show that \( u \) is the unique viscosity solution, and conclude that the entire sequence must converge uniformly to \( u \).]

3. Suppose that \( S_h \) depends only on the forward and backward neighboring grid points in each direction, so that we can write

\[ S_h(u, u(x), x) = F(\nabla_1^- u(x), -\nabla_1^+ u(x), \ldots, \nabla_n^- u(x), -\nabla_n^+ u(x), u(x), x). \]

Let us set \( F = F(a_1, \ldots, a_{2n}, z, x) \). You may assume that \( H \) and \( F \) are smooth.

(a) Show that \( S_h \) is monotone if and only if \( F_{a_i} \geq 0 \) for all \( i \).

(b) Show that \( S_h \) is consistent if and only if

\[ F(p_1, -p_2, \ldots, -p_n, -p, z, x) = H(p, z, x) \]

for all \( p \in \mathbb{R}^n, z \in \mathbb{R} \text{ and } x \in [0,1]^n_h \).

(c) Find a monotone and consistent scheme for the linear PDE

\[ a_1 u_{x_1} + \cdots + a_n u_{x_n} = f(x), \]

where \( a_1, \ldots, a_n \) are real numbers. Compare your scheme with the direction of the projected characteristics. [Hint: Your solution should depend on the signs of the \( a_i \).]
(d) Suppose that $H$ is Lipschitz continuous and define

$$a := \sup \{|D_p H(p, z, x)| : p \in \mathbb{R}^n, z \in \mathbb{R}, x \in [0, 1]^n\}.$$

The Lax-Friedrichs scheme is defined by

$$S_h(u, u(x), x) := H(\nabla_h u(x), u(x), x) - \frac{ah}{2} \Delta_h u(x),$$

where

$$\nabla_h u(x) := \left(\frac{u(x + h e_1) - u(x - h e_1)}{2h}, \ldots, \frac{u(x + h e_n) - u(x - h e_n)}{2h}\right),$$

and

$$\Delta_h u(x) := \sum_{i=1}^n \frac{u(x + h e_i) - 2u(x) + u(x - h e_i)}{h^2}.$$

Show that the Lax-Friedrichs scheme is monotone and consistent. [Hint: Rewrite the scheme as a function of the forward and backward differences $\nabla^\pm_h u(x)$, as above.]

4. Let $U := B^0(0, 1)$ and $\varepsilon > 0$. Consider the nonlocal integral equation

$$(I_{\varepsilon}) \begin{cases} (1 + c\varepsilon^2) u_\varepsilon(x) - \int_{B(x, \varepsilon)} u_\varepsilon \, dy = c\varepsilon^2 f(x) & \text{if } x \in U \\ u_\varepsilon(x) = 0 & \text{if } x \in \Gamma_\varepsilon, \end{cases}$$

where $c = \frac{1}{2(n+2)}$, $u_\varepsilon : \Gamma_\varepsilon \cup U \to \mathbb{R}$, $f \in C(U)$, and

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n \setminus U : \text{dist}(x, \partial U) \leq \varepsilon\}.$$ 

Follow the steps below to show that as $\varepsilon \to 0^+$, $u_\varepsilon$ converges uniformly to the viscosity solution $u$ of

$$(P) \begin{cases} u - \Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

The proof is based on recognizing $(I_{\varepsilon})$ as a monotone approximation scheme for $(P)$. Unless otherwise specified, any function $u : U \to \mathbb{R}$ is implicitly extended to be identically zero on $\Gamma_\varepsilon$.

(a) Show that there exists a unique function $u_\varepsilon \in C(\overline{U})$ solving $(I_{\varepsilon})$. [Hint: Show that the mapping $T : C(\overline{U}) \to C(\overline{U})$ defined by

$$T[u](x) := \frac{1}{1 + c\varepsilon^2} \int_{B(x, \varepsilon)} u \, dy + \frac{c\varepsilon^2}{1 + c\varepsilon^2} f(x)$$

is a contraction mapping. Use the usual norm $||u|| := \max\overline{U} |u|$ on $C(\overline{U})$. Then appeal to Banach’s fixed point theorem.]
(b) Define \( S_\varepsilon : L^\infty(U \cup \Gamma_\varepsilon) \times \mathbb{R} \times U \to \mathbb{R} \) by
\[
S_\varepsilon(u, t, x) := (1 + c\varepsilon^2)t - \int_{B(x, \varepsilon)} u \, dy.
\]
Show that \( S_\varepsilon \) is monotone, i.e., for all \( t \in \mathbb{R}, x \in U \), and \( u, v \in L^\infty(U \cup \Gamma_\varepsilon) \)
\[
u \leq v \text{ on } B(x, \varepsilon) \implies S_\varepsilon(u, t, x) \geq S_\varepsilon(v, t, x).
\]

(c) Show that the following comparison principle holds: Let \( u, v \in L^\infty(U \cup \Gamma_\varepsilon) \) such that \( u|_{\Gamma}, v|_{\Gamma} \in C(\overline{U}) \). If \( u \leq v \) on \( \Gamma_\varepsilon \) and \( S_\varepsilon(u, u(x), x) \leq S_\varepsilon(v, v(x), x) \) at all \( x \in U \), then \( u \leq v \) on \( U \).

(d) Use the comparison principle to show that there exists \( C > 0 \) such that
\[
|u_\varepsilon(x)| \leq C(1 + 3\varepsilon - |x|^2),
\]
for all \( x \in U \) and \( 0 < \varepsilon \leq 1 \), where \( C \) depends only on \( \|f\| = \max_{\overline{U}} |f| \). [Hint: Compare against \( v(x) := C(1 + 3\varepsilon - |x|^2) \) and \( -v \), and adjust the constant \( C \) appropriately.]

(e) Use the method of weak upper and lower limits to show that \( u_\varepsilon \to u \) uniformly on \( U \), where \( u \) is the viscosity solution of (P). You may assume a comparison principle holds for (P) for semicontinuous viscosity solutions. That is, if \( u \in \text{USC}(\overline{U}) \) is a viscosity subsolution of (P) and \( v \in \text{LSC}(\overline{U}) \) is a viscosity supersolution, and \( u \leq v \) on \( \partial U \), then \( u \leq v \) in \( U \). [Hint: You will find the identity in the hint from HW1 Problem 4 useful.]