Math 8590 – Homework 4 Solutions

1. Implement both fast sweeping and fast marching in the programming language of your choice for solving the one dimensional eikonal equation

$$|u'(x)| = f(x) \text{ for } x \in (0,1),$$

with boundary conditions $u(0) = u_0$ and $u(1) = u_1$. Experiment with different funtions $f \geq 0$ and different boundary conditions. Are the boundary conditions always attained continuously? Which method is faster? Produce some plots of solutions to hand in, and turn in a working version of your code.

2. Suppose the numerical solutions u_h of our monotone scheme $S_h = 0$ are uniformly Lipschitz continuous, i.e., there exists $C > 0$ such that

$$|u_h(x) - u_h(y)| \leq C|x - y| \quad \text{for all } x, y \in [0,1]^n_h \text{ and } h > 0.$$

This is a stronger form of stability. Prove the Barles-Souganidis convergence theorem from class without assuming **strong uniqueness**. You can assume that ordinary uniqueness holds, that is, there is at most one viscosity solution satisfying the boundary conditions in the usual sense. [Hint: Use the Arzelà-Ascoli Theorem to extract a subsequence u_{h_k} converging uniformly to a continuous function $u \in C([0,1]^n)$. Show that u is the unique viscosity solution, and conclude that the entire sequence must converge uniformly to u.]

Proof. We briefly sketch the proof. By a minor extension of Arzela-Ascoli, every subsequence u_{h_j} contains a further subsequence converging uniformly to a continuous function $u \in C(\overline{U})$. Then use the standard viscosity machinery and monotonicity of the scheme to show that u is the unique viscosity solution of the scheme. Now, assume by way of contradiction that the entire sequence u_h does not converge uniformly to u. Then we can extract a subsequence u_{h_j} for which $\sup_{[0,1]^n_{h_j}} |u(x) - u_{h_j}(x)| \geq \delta > 0$ as $h_j \to 0$. But then, as above, u_{h_j} contains a subsequence converging uniformly to u, which is a contradiction. Note the proof only requres uniqueness of continuous viscosity solutions of the limiting PDE. □

3. Suppose that S_h depends only on the forward and backward neighboring grid points in each direction, so that we can write

$$S_h(u, u(x), x) = F(\nabla_1^- u(x), -\nabla_1^+ u(x), \ldots, \nabla_n^- u(x), -\nabla_n^+ u(x), u(x), x).$$

Let us set $F = F(a_1, \ldots, a_{2n}, z, x)$. You may assume that H and F are smooth.

(a) Show that S_h is monotone if and only if $F_{a_i} \geq 0$ for all i.

Solution. The statement is obvious from the identity

$$S_h(u,t,x) = F\Big(\frac{t - u(x - he_1)}{h}, \frac{t - u(x + he_1)}{h}, \ldots, \frac{t - u(x - he_n)}{h}, \frac{t - u(x + he_n)}{h}, t, x\Big).$$

□

(b) Show that S_h is consistent if and only if

$$F(p_1, -p_2, \ldots, p_n, -p_n, z, x) = H(p, z, x)$$

for all $p \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $x \in [0,1]_h^n$.

Solution. Let $\varphi \in C^\infty(\mathbb{R}^n)$, $x \in (0,1)^n$, and set $p = D\varphi(x)$. Since F is smooth,

$$\lim_{\substack{y \to x \\ h \to 0^+ \\ \gamma \to 0}} S_h(\varphi + \gamma, \varphi(y) + \gamma, y) = H(p_1, -p_1, \ldots, p_n, -p_n, \varphi(x), x).$$

The result immediately follows. □

(c) Find a monotone and consistent scheme for the linear PDE

$$a_1 u_{x_1} + \cdots + a_n u_{x_n} = f(x),$$

where $a_1, \ldots, a_n$ are real numbers. Compare your scheme with the direction of the projected characteristics. [Hint: Your solution should depend on the signs of the a_i.]

Solution. As discovered in part (a), monotone schemes are increasing functions of $\nabla_i^- u(x)$ and decreasing functions of $\nabla_i^+ u(x)$. Thus, if a_i is positive, we should select backward differences, and if a_i is negative, then we should select forward differences. Let

$$m_i := \begin{cases} 1, & \text{if } a_i \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

We can write a monotone scheme as

$$S_h(u, u(x), x) = \sum_{i=1}^n m_i \nabla_i^- u(x) + (1 - m_i) \nabla_i^+ u(x).$$

Notice that the characteristics flow in the direction

$$\dot{x}(s) = D_p H(p(s), z(s), x(s)) = (a_1, \ldots, a_n).$$

When the characteristics are flowing in the positive x_i direction, we say the 'wind' is blowing from the left to the right (by 'wind', we mean information is propagating in this direction). In this case $a_i \geq 0$ and we choose backward differences $\nabla_i^- u(x)$. This is called 'upwinding', and reflects the fact that $u(x)$ should depend on the values of u in the direction from which the wind is blowing. When the characteristics flow in the negative x_i direction, so $a_i < 0$, the wind is blowing from the right to the left, and we choose forward differences. Again, this reflects the fact that the solution $u(x)$ depends on the values of u in the direction from which the wind is blowing. These heuristics are why monotone schemes for first order equations are called upwind schemes. □

(d) Suppose that H is Lipschitz continuous and define

$$a := \sup\{|D_p H(p, z, x)| \,:\, p \in \mathbb{R}^n,\ z \in \mathbb{R},\ x \in [0,1]^n\}.$$

The Lax-Friedrichs scheme is defined by

$$S_h(u, u(x), x) := H\left(\nabla_h u(x), u(x), x\right) - \frac{ah}{2}\Delta_h u(x),$$

where

$$\nabla_h u(x) := \left(\frac{u(x+he_1) - u(x-he_1)}{2h}, \ldots, \frac{u(x+he_n) - u(x-he_n)}{2h}\right),$$

and

$$\Delta_h u(x) := \sum_{i=1}^n \frac{u(x+he_i) - 2u(x) + u(x-he_i)}{h^2}.$$

Show that the Lax-Friedrichs scheme is monotone and consistent. [Hint: Rewrite the scheme as a function of the forward and backward differences $\nabla_i^{\pm} u(x)$, as above.]

Solution. Write $a \in \mathbb{R}^{2n}$ as $a = (b_1, c_1, \ldots, b_n, c_n)$. Then we can write the Lax-Friedrichs scheme as

$$F(a, z, x) = H\left(\frac{b-c}{2}, z, x\right) + \frac{a}{2}\sum_{i=1}^n (b_i + c_i).$$

Then

$$F_{b_i} = \frac{1}{2}H_{p_i}\left(\frac{b-c}{2}, z, x\right) + \frac{a}{2} \geq 0,$$

and

$$F_{c_i} = -\frac{1}{2}H_{p_i}\left(\frac{b-c}{2}, z, x\right) + \frac{a}{2} \geq 0.$$

Therefore F is monotone. When $b = p$ and $c = -p$ we have

$$F(p_1, -p_1, \ldots, p_n, -p_n, z, x) = H(p, z, x).$$

Therefore F is consistent. □

4. Let $U := B^0(0,1)$ and $\varepsilon > 0$. Consider the nonlocal integral equation

$$(\mathrm{I}_\varepsilon) \quad \begin{cases} (1 + c\varepsilon^2)u_\varepsilon(x) - \fint_{B(x,\varepsilon)} u_\varepsilon\, dy = c\varepsilon^2 f(x) & \text{if } x \in U \\ u_\varepsilon(x) = 0 & \text{if } x \in \Gamma_\varepsilon, \end{cases}$$

where $c = \frac{1}{2(n+2)}$, $u_\varepsilon : \Gamma_\varepsilon \cup U \to \mathbb{R}$, $f \in C(\overline{U})$, and

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n \setminus U \,:\, \mathrm{dist}(x, \partial U) \leq \varepsilon\}.$$

Follow the steps below to show that as $\varepsilon \to 0^+$, u_ε converges uniformly to the viscosity solution u of

$$\text{(P)} \quad \begin{cases} u - \Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

The proof is based on recognizing (I_ε) as a monotone approximation scheme for (P). Unless otherwise specified, any function $u : U \to \mathbb{R}$ is implicitly extended to be identically zero on Γ_ε.

(a) Show that there exists a unique function $u_\varepsilon \in C(\overline{U})$ solving (I_ε). [Hint: Show that the mapping $T : C(\overline{U}) \to C(\overline{U})$ defined by

$$T[u](x) := \frac{1}{1 + c\varepsilon^2} \fint_{B(x,\varepsilon)} u\, dy + \frac{c\varepsilon^2}{1 + c\varepsilon^2} f(x)$$

is a contraction mapping. Use the usual norm $\|u\| := \max_{\overline{U}} |u|$ on $C(\overline{U})$. Then appeal to Banach's fixed point theorem.]

Solution. Define $T : C(\overline{U}) \to C(\overline{U})$ by

$$T[u](x) := \frac{1}{1 + c\varepsilon^2} \fint_{B(x,\varepsilon)} u\, dy + \frac{c\varepsilon^2}{1 + c\varepsilon^2} f(x).$$

Since $x \mapsto \fint_{B(x,\varepsilon)} u\, dy$ is continuous provided $u \in L^\infty_{loc}(\mathbb{R}^n)$, the mapping T is well-defined. We claim that T is a contraction. Let $\alpha = \frac{1}{1+c\varepsilon^2}$, and fix $u, v \in C(\overline{U})$. Then

$$\begin{aligned} |T[u](x) - T[v](x)| &= \left| \alpha \fint_{B(x,\varepsilon)} u(y) - v(y)\, dy \right| \\ &\leq \frac{\alpha}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)\cap U} |u(y) - v(y)|\, dy \\ &\leq \alpha \frac{|B(x,\varepsilon) \cap U|}{|B(x,\varepsilon)|} \|u - v\| \leq \alpha \|u - v\|. \end{aligned}$$

Therefore

$$\|T[u] - T[v]\| \leq \alpha \|u - v\|,$$

for all $u, v \in C(\overline{U})$, where $0 < \alpha < 1$. Therefore T is a contraction mapping, and by Banach's fixed point theorem, there exists a unique $u_\varepsilon \in C(\overline{U})$ such that $T[u_\varepsilon] = u_\varepsilon$, i.e., u_ε is the unique solution of (I_ε). □

(b) Define $S_\varepsilon : L^\infty(U \cup \Gamma_\varepsilon) \times \mathbb{R} \times U \to \mathbb{R}$ by

$$S_\varepsilon(u, t, x) := (1 + c\varepsilon^2)t - \fint_{B(x,\varepsilon)} u\, dy.$$

Show that S_ε is monotone, i.e., for all $t \in \mathbb{R}$, $x \in U$, and $u, v \in L^\infty(U \cup \Gamma_\varepsilon)$

$$u \leq v \text{ on } B(x,\varepsilon) \implies S_\varepsilon(u, t, x) \geq S_\varepsilon(v, t, x).$$

Solution. The monotonicity is immediate, since $u \le v \implies \fint_{B(x,\varepsilon)} u\,dy \le \fint_{B(x,\varepsilon)} v\,dy$ for all x. $\square$

(c) Show that the following comparison principle holds: Let $u, v \in L^\infty(U \cup \Gamma_\varepsilon)$ such that $u|_{\overline{U}}, v|_{\overline{U}} \in C(\overline{U})$. If $u \le v$ on Γ_ε and $S_\varepsilon(u, u(x), x) \le S_\varepsilon(v, v(x), x)$ at all $x \in U$, then $u \le v$ on U.

Solution. Set $w := u - v$, and note that since S is linear we have

$$S_\varepsilon(w, w(x), x) = S_\varepsilon(u, u(x), x) - S_\varepsilon(v, v(x), x) \le 0 \quad \text{for all } x \in U.$$

Therefore

$$(1 + c\varepsilon^2) w(x) \le \fint_{B(x,\varepsilon)} w(y)\,dy \quad \text{for all } x \in \overline{U},$$

due to the fact that w is uniformly continuous on U.

Assume to the contrary that $w(x) > 0$ for some $x \in U$. Let $x_0 \in \overline{U}$ be a point at which w attains its positive maximum over $\overline{U}$. Since $w \le 0$ on Γ_ε, $w(x_0) \ge w(x)$ for all $x \in U \cup \Gamma_\varepsilon$. Therefore

$$(1 + c\varepsilon^2) w(x_0) \le \fint_{B(x_0,\varepsilon)} w(y)\,dy \le w(x_0),$$

which is a contradiction. Therefore $w \le 0$ on U, and hence $u \le v$ on U. $\square$

(d) Use the comparison principle to show that there exists $C > 0$ such that

$$|u_\varepsilon(x)| \le C(1 + 3\varepsilon - |x|^2),$$

for all $x \in U$ and $0 < \varepsilon \le 1$, where C depends only on $\|f\| = \max_{\overline{U}} |f|$. [Hint: Compare against $v(x) := C(1 + 3\varepsilon - |x|^2)$ and $-v$, and adjust the constant C appropriately.]

Solution. For $C > 0$ to be determined later, set $v(x) := C(1 + 3\varepsilon - |x|^2)$. A computation yields

$$S_\varepsilon(v, v(x), x) = c\varepsilon^2 v(x) + 2nCc\varepsilon^2.$$

Note that for $x \in U \cup \Gamma_\varepsilon$, $|x| \le 1 + \varepsilon$ and so

$$v(x) \ge C(1 + 3\varepsilon - (1+\varepsilon)^2) = C(3\varepsilon - 2\varepsilon - \varepsilon^2) \ge 0,$$

for $0 < \varepsilon \le 1$. Therefore $v \ge 0$ on $U \cup \Gamma_\varepsilon$ and

$$S_\varepsilon(v, v(x), x) \ge 2nCc\varepsilon^2.$$

By choosing $C > 0$ large enough, depending only on $\|f\|$, we have

$$S_\varepsilon(v, v(x), x) \ge c\varepsilon^2 f(x) = S_\varepsilon(u_\varepsilon, u_\varepsilon(x), x)$$

for all $x \in U$. Since $u_\varepsilon = 0 \le v$ on Γ_ε, the comparison principle from problem 3 shows that $u_\varepsilon \le v$ on U. A similar argument shows that $u_\varepsilon \ge -v$. $\square$

(e) Use the method of weak upper and lower limits to show that $u_\varepsilon \to u$ uniformly on $\overline{U}$, where u is the viscosity solution of (P). You may assume a comparison principle holds for (P) for semicontinuous viscosity solutions. That is, if $u \in \mathrm{USC}(\overline{U})$ is a viscosity subsolution of (P) and $v \in \mathrm{LSC}(\overline{U})$ is a viscosity supersolution, and $u \le v$ on ∂U, then $u \le v$ in U. [Hint: You will find the identity in the hint from HW1 Problem 4 useful.]

Solution. We consider the weak upper and lower limits

$$\overline{u}(x) := \limsup_{(y,\varepsilon)\to(x,0^+)} u_\varepsilon(y) \quad \text{and} \quad \underline{u}(x) := \liminf_{(y,\varepsilon)\to(x,0^+)} u_\varepsilon(y).$$

By problem 4, these are bounded functions on $\overline{U}$, and we showed in class that $\overline{u} \in \mathrm{USC}(\overline{U})$ and $\underline{u} \in \mathrm{LSC}(\overline{U})$. Furthermore, the estimate in problem 4 also shows that $\overline{u} = \underline{u} = 0$ on $\partial U = \partial B(0,1)$.

To complete the proof, we just need to show that $\overline{u}$ is a viscosity subsolution of (P), and $\underline{u}$ is a viscosity supersolution of (P). Then the assumed comparison principle gives $\overline{u} \le \underline{u}$, and so $\overline{u} = \underline{u} = u$, where u is the unique viscosity solution of (P).

We'll show that $\overline{u}$ is a viscosity subsolution of (P); the proof that $\underline{u}$ is a supersolution is similar. We first extend $\overline{u}(x) = 0$ for $x \notin \overline{U}$. Let $x_0 \in U$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\overline{u} - \varphi$ has a local maximum at x. As usual, we may assume that $\overline{u} - \varphi$ actually has a strict global maximum at x_0 over $\mathbb{R}^n$ and $\varphi(x_0) = \overline{u}(x_0)$. Then there exist sequences $\varepsilon_k \to 0^+$ and $x_k \to x_0$ such that $u_{\varepsilon_k}(x_k) \to \overline{u}(x_0)$ and $u_{\varepsilon_k} - \varphi$ has a global maximum at x_k for each k. Define

$$\varphi_k(x) := \varphi(x) + \gamma_k,$$

where $\gamma_k := u_{\varepsilon_k}(x_k) - \varphi(x_k)$. Then

$$\varphi_k(x_k) = u_{\varepsilon_k}(x_k) \quad \text{and} \quad u_{\varepsilon_k} \le \varphi_k \quad \text{on } U \cup \Gamma_{\varepsilon_k}.$$

By the monotonicity of the scheme S_ε we have

$$c\varepsilon_k^2 f(x_k) = S_{\varepsilon_k}(u_{\varepsilon_k}, u_{\varepsilon_k}(x_k), x_k) \ge S_{\varepsilon_k}(\varphi_k, \varphi_k(x_k), x_k).$$

Unwrapping the definition of S_ε we have

$$(1 + c\varepsilon_k^2)\varphi_k(x_k) - \fint_{B(x_k,\varepsilon_k)} \varphi_k(y)\,dy \le c\varepsilon_k^2 f(x_k).$$

Since $\varphi_k = \varphi + \gamma_k$ we have

$$\varphi(x_k) + 2(n+1)\fint_{B(x_k,\varepsilon_k)} \frac{\varphi(x_k) - \varphi(y)}{\varepsilon_k^2}\,dy + \gamma_k \le f(x_k).$$

Sending $k \to \infty$, recalling $\gamma_k \to 0$, and using the identity from Homework 8, Problem 4, we have

$$\overline{u}(x_0) - \Delta\varphi(x_0) \le f(x_0).$$

Therefore $\overline{u}$ is a viscosity subsolution of (P). □