Math 8590: Viscosity Solutions

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Notation

Notation follows Evans PDE book:

- $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$
- $u_{x_i}(x) = \lim_{h \to 0} \frac{u(x+he_i) - u(x)}{h}$
- $Du = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$
- $D^2 u = (u_{x_i x_j})_{i,j}$
- $C^k(U) = \{k\text{-times continuously differentiable } u : U \to \mathbb{R}\}$
- $C^k(\overline{U}) = \{u \in C^k(U) \,:\, D^\alpha u \text{ uniformly continuous, } |\alpha| \le k\}^*$

  * On bounded subsets of U if unbounded.
- $C^{k,\alpha}(\overline{U})$ are the Hölder spaces…

Introduction

We are concerned with fully nonlinear 1st and 2nd order partial differential equations (PDE)

$$F(D^2u, Du, u, x) = 0. \tag{1}$$

The PDE will hold in some domain $U \subset \mathbb{R}^n$ with some appropriate boundary condition.

Definition 1. A *classical* solution is a function $u \in C^2(U)$ such that (1) is satisfied at each $x \in U$.

Need for a nonsmooth (weak) solution

For many important applications, classical solutions do not exist.

- Optimal control theory (including stochastic versions)
- Differential games
- Calculus of variations (also in L^∞)
- Geometric evolutions (e.g., curvature motion, level-set method)
- Computer vision and image processing
- More recently machine learning

Viscosity solution is a notion of weak solution for fully nonlinear PDEs that provides the physically correct nonsmooth solution to all the problems above (and, in general, most problems*).

Example 1 (Distance function)**.** Let Γ be a closed subset of $\mathbb{R}^n$ and let $u : \mathbb{R}^n \to [0, \infty)$ be the *distance function to* Γ, defined by

$$u(x) = \operatorname{dist}(x, \Gamma) := \min_{y \in \Gamma} |x - y|. \tag{2}$$

Optimal control theory (Evans Chapter 10)

We have a state $\mathbf{x}(t) \in \mathbb{R}^n$ that evolves according to the dynamics

$$
\begin{cases} \dot{\mathbf{x}}(s) = \mathbf{f}(\mathbf{x}(s), \boldsymbol{\alpha}(s)), \quad (t < s < T) \\ \mathbf{x}(t) = x. \end{cases} \tag{3}
$$

The goal is to select the control $\boldsymbol{\alpha}(t)$ so as to minimize the cost functional

$$
C_{x,t}[\boldsymbol{\alpha}(\cdot)] := \int_t^T r(\mathbf{x}(s), \boldsymbol{\alpha}(s))\, ds + g(\mathbf{x}(T)). \tag{4}
$$

The *value function*

$$
u(x,t) = \inf_{\boldsymbol{\alpha}(\cdot)} C_{x,t}[\boldsymbol{\alpha}(\cdot)] \tag{5}
$$

satisfies (in the viscosity sense) the Hamilton-Jacobi-Bellman equation

$$
\begin{cases} u_t + \min_a \{\mathbf{f}(x,a) \cdot Du + r(x,a)\} = 0 \quad \text{in } \mathbb{R}^n \times (0,T) \\ u = g \quad \text{on } \mathbb{R}^n \times \{t = T\}. \end{cases} \tag{6}
$$

Example 2 (Mean curvature motion)

For a smooth curve $\gamma(p)$, $p \in \mathbb{R}^2$, the curvature at p is defined as

$$\kappa(p) = \frac{d\theta}{ds}(p), \tag{7}$$

where $s =$ arclength and $\theta =$ angle between tangent and reference axis.

Exercise 1. The curvature of a circle of radius R is $\kappa = 1/R$. Here, R is the *radius of curvature.*

Exercise 2. For a curve $\gamma(\tau) = (x(\tau), y(\tau))$, the curvature is

$$\kappa(t) = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}}, \tag{8}$$

where $' = \frac{d}{d\tau}$.

Exercise 3. For a curve $\gamma(x) = (x, f(x))$ with $f(0) = f'(0) = 0$, the curvature at $(0,0)$ is

$$\kappa = f''(0). \tag{9}$$

Curvature motion of planar curves

Curvature motion moves a curve in the direction of its inward normal with a speed equal to curvature. That is, curvature motion generates a family of curves $C(t,\tau) = (x(t,\tau), y(t,\tau))$ satisfying the coupled PDE

$$\begin{cases} \dfrac{\partial x}{\partial t} = \left(\dfrac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}} \right) \dfrac{-y'}{\sqrt{x'^2 + y'^2}}, \\ \dfrac{\partial y}{\partial t} = \left(\dfrac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}} \right) \dfrac{x'}{\sqrt{x'^2 + y'^2}}. \end{cases} \tag{10}$$

In more compact notation

$$\frac{\partial C}{\partial t} = \kappa \mathbf{N},$$

where $\mathbf{N} = (-y', x')/\sqrt{x'^2 + y'^2}$ is the unit inward normal vector to C.

Curvature motion is gradient descent on length

Define the length of $C = (x(\tau), y(\tau))$ by

$$L(C) = \int_a^b \sqrt{x'^2 + y'^2}\, d\tau. \tag{11}$$

Suppose $C(s) = (x(s), y(s))$ is parameterised by arclength s, and consider a perturbation in the normal direction $C(s) + \varepsilon v(s)\mathbf{N}(s)$.

We compute $(\,\dot{}\, = \frac{d}{ds})$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L(C + \varepsilon v \mathbf{N}) = \int_a^b (\ddot{x}\dot{y} - \dot{x}\ddot{y}) v\, ds. \tag{12}$$

Gage-Hamilton-Grayson Theorem

A smooth simple closed curve in the **plane** that undergoes curvature motion remains smoothly embedded without self-intersections, will eventually become and remain convex, and shrink to a single point, becoming asympototically *round*.

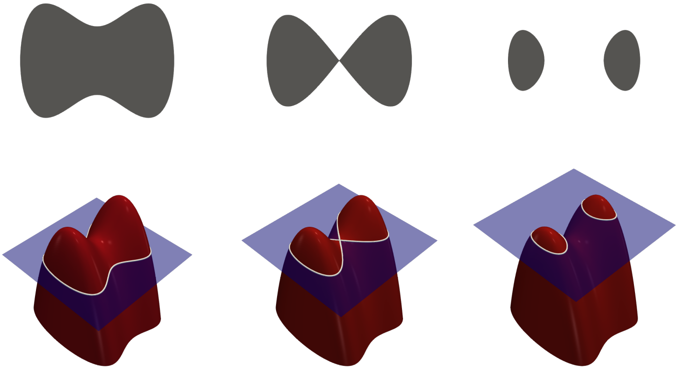
- Gage & Hamilton (1986) proved smooth convex curves contract to a point.
- Grayson (1987) proved that every non-convex curve eventually becomes convex.
- Simpler proofs have emerged since (Andrews & Bryan (2011)).

This is all in the **classical setting** (solutions are smooth, etc).

Level-set method

The level-set method represents the evolving curve $C(t)$ implicitly as the zero level-set of a function $u : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$. That is,

$$C(t) = \{x \in \mathbb{R}^2 \,:\, u(x,t) = 0\}. \tag{13}$$



https://en.wikipedia.org/wiki/Level-set_method

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$$C(t) = \{x \in \mathbb{R}^2 \, : \, u(x,t) = 0\}. \tag{14}$$

If $C(t)$ evolves according to curvature motion $\partial C/\partial t = \kappa \mathbf{N}$ then u satisfies (formally) the level-set equation

$$u_t - \frac{\nabla^{\perp} u \cdot D^2 u \nabla^{\perp} u}{|\nabla u|^2} = 0, \tag{15}$$

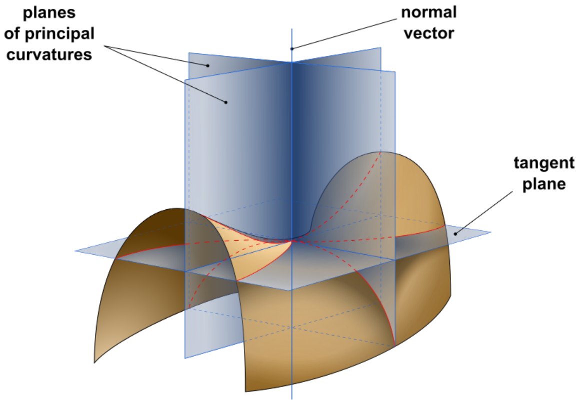
where $\nabla^{\perp} u = (u_{x_2}, -u_{x_1})$. The **viscosity solution** theory applies to the level-set equation for curvature motion (15). The PDE can be rewritten as

$$u_t - |\nabla u| \, \mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right) = 0. \tag{16}$$

Curvature of surfaces

For a smooth surface $S \subset \mathbb{R}^3$ we have curvatures in each direction $v \in T_pS$:

$$\kappa(p; v) = \text{Curvature of interesection of } S \text{ and the plane span}(v, N)$$



https://en.wikipedia.org/wiki/Principal_curvature

Curvature of surfaces

The *principal curvatures* are

$$\kappa_1(p) = \min_{v \in T_pS} \kappa(p;v) \quad \text{and} \quad \kappa_2(p) = \max_{v \in T_pS} \kappa(p;v).$$

The principal curvatures occur in orthogonal directions if $\kappa_1 \neq \kappa_2$.

We define the *mean curvature*

$$H(p) = \kappa_1(p) + \kappa_2(p), \tag{17}$$

and the *Gauss curvature*

$$K(p) = \kappa_1(p)\kappa_2(p). \tag{18}$$

Gauss curvature is an *intrinsic* quantity (Gauss's Theorema Egregium).

Curvature of surfaces

Exercise 4. For a flat space (e.g., a plane), $\kappa_1 = \kappa_2 = H = K = 0$.

Exercise 5. For a sphere of radius $R > 0$, $\kappa_1 = \kappa_2 = 1/R$, $H = 2/R$ and $K = 1/R^2$.

Exercise 6. For a cylinder of radius $R > 0$, $\kappa_1 = 0$, $\kappa_2 = 1/R$, $H = 1/R$ and $K = 0$.

Curvature of surfaces

Exercise 7. For a surface $z = f(x, y)$ with $f(0, 0) = 0$ and $Df(0, 0) = 0$,

$$\kappa_1(0,0), \kappa_2(0,0) = \text{Eigenvalues of Hessian matrix } D^2 f(0,0),$$

$$\text{(Mean curvature) } H(0,0) = \text{Trace}(D^2 f(0,0)) = f_{xx}(0,0) + f_{yy}(0,0),$$

$$\text{(Gauss curvature) } K(0,0) = \det(D^2 f(0,0)) = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}(0,0)^2.$$

Mean curvature motion

Mean curvature motion evolves a surface with normal speed equal to mean curvature:

$$\frac{\partial S}{\partial t}(p) = H(p)\mathbf{N}(p). \tag{19}$$

Mean curvature motion is gradient descent on the surface area functional

$$A(S) = \int_S dS. \tag{20}$$

Demo: Gage-Hamilton-Grayson theorem does **not** hold in dimension $n \geq 3$. Surface can develop singluarities in finite time, after which point **classical** solutions fail to exist.

Level-set method

The level-set method represents the evolving surface $S(t)$ implicitly as the zero level-set of a function $u : \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}$. That is,

$$S(t) = \{x \in \mathbb{R}^3 \; : \; u(x,t) = 0\}. \tag{21}$$

If $S(t)$ evolves according to mean curvature motion $\partial S/\partial t = H\mathbf{N}$ then u satisfies (formally) the level-set equation

$$u_t - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0. \tag{22}$$

The viscosity solution of (22) exists and is unique for all time, allowing us to interpret mean curvature motion beyond singularities.

Level-set method

References:

- Level-set method was invented by Sethian and Osher (1988) as an efficient numerical scheme for tracking evolving fronts and surfaces.
- Evans and Spruck (1991) proved well-posedness of the level-set equation for mean curvature motion in the viscosity sense, and proposed it as a notion of generalized mean curvature motion.