Math 8590: Viscosity Solutions
Second order equations

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Second order equations

We consider in this section the comparison principle for viscosity solutions of second order equations

$$F(D^2u, Du, u, x) = 0 \quad \text{in } U, \tag{1}$$

where $U \subset \mathbb{R}^n$, and F is degenerate elliptic and monotone.

Jensen's Lemma

Lemma 1 (Jensen's Lemma)**.** *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be semiconvex and let x_0 be a strict local maximum of φ. For $p \in \mathbb{R}^n$ set $\varphi_p(x) = \varphi(x) - p \cdot (x - x_0)$. Then for $r > 0$ sufficiently small and all $\delta > 0$ the set*

$$K = \{y \in B(x_0, r) : \exists p \in B(0, \delta) \text{ such that } \varphi_p(x) \leq \varphi_p(y) \text{ for } x \in B(x_0, r)\}$$

has positive measure.

Jensen's Lemma

Lemma 2 (Jensen's Lemma)**.** *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be semiconvex and let x_0 be a strict local maximum of φ. For $p \in \mathbb{R}^n$ set $\varphi_p(x) = \varphi(x) - p \cdot (x - x_0)$. Then for $r > 0$ sufficiently small and all $\delta > 0$ the set*

$$K = \{y \in B(x_0, r) : \exists p \in B(0, \delta) \textit{ such that } \varphi_p(x) \leq \varphi_p(y) \textit{ for } x \in B(x_0, r)\}$$

has positive measure.

Proposition 1. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be semiconvex and let x_0 be a local maximum of φ. Then there exists $x_k \to x_0$ such that φ is twice differentiable at x_k, $D\varphi(x_k) \to 0$ as $k \to \infty$ and $D^2\varphi(x_k) \leq \varepsilon_k I$ for a sequence $\varepsilon_k \to 0$.*

Semiconvex comparison

We assume $U \subset \mathbb{R}^n$ is open and bounded, F is continuous, degenerate elliptic, and monotone, and satisfies

$$F(X,p,z,y) - F(X,p,z,x) \leq \omega(|x-y|(1+|p|)) \tag{2}$$

for all $x, y \in U$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$, and symmetric matrices X, where ω is a modulus of continuity.

Lemma 3 (Semiconvex comparison). *Let $u \in C(\overline{U})$ be a semiconvex viscosity solution of*

$$F(D^2u, Du, u, x) \leq 0 \quad \text{in } U,$$

and let $v \in C(\overline{U})$ be a semiconcave viscosity solution of

$$F(D^2v, Dv, v, x) - a \geq 0 \quad \text{in } U,$$

for some $a > 0$. If $u \leq v$ on ∂U then $u \leq v$ in U.

Continuous comparison

We now assume F has the form

$$F(X, p, z, x) = \lambda z + H(X, p) - f(x), \tag{3}$$

where $\lambda \geq 0$.

Theorem 1 (Continuous comparison)**.** *Let $u \in C(\overline{U})$ be a viscosity subsolution of*

$$F(D^2 u, Du, u, x) \leq 0 \quad \text{in } U,$$

and let $v \in C(\overline{U})$ be a viscosity solution of

$$F(D^2 v, Dv, v, x) - a \geq 0 \quad \text{in } U,$$

for some $a > 0$. If $u \leq v$ on ∂U then $u \leq v$ in U.

Subjets and superjets

Definition 1. Let $\mathcal{O} \subset \mathbb{R}^n$, $u : \mathcal{O} \to \mathbb{R}$, and $x_0 \in \mathcal{O}$. The superjet $J^{2,+}_{\mathcal{O}} u(x_0)$ is defined as the set of all $(p, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ for which

$$u(x) \leq u(x_0) + p \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T X (x - x_0) + o(|x - x_0|^2)$$

as $\mathcal{O} \ni x \to x_0$.

Similarly, the subjet $J^{2,-}_{\mathcal{O}} u(x_0)$ is defined as the set of all $(p, X) \in \mathbb{R}^n \times \mathcal{S}(n)$ for which

$$u(x) \geq u(x_0) + p \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T X (x - x_0) + o(|x - x_0|^2)$$

as $\mathcal{O} \ni x \to x_0$.

Note: If $x \in \text{Interior}(\mathcal{O})$ then we write just $J^{2,+}u(x)$ and $J^{2,-}u(x)$, since the domain is unimportant.

Subjets and superjets

Example 1. Define $u : \mathbb{R} \to \mathbb{R}$ by

$$u(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ ax + \frac{b}{2}x^2, & \text{if } x \geq 0. \end{cases}$$

Then $J^{2,+}_{[-1,0]}u(0) = ((-\infty, 0) \times \mathbb{R}) \cup (\{0\} \times [0, \infty))$, while

$$J^{2,+}_{\mathbb{R}}u(0) = \begin{cases} \varnothing, & \text{if } a > 0 \\ \{0\} \times [\max\{0, b\}, \infty), & \text{if } a = 0 \\ ((a, 0) \times \mathbb{R}) \cup (\{0\} \times [0, \infty)) \cup (\{a\} \times [b, \infty)), & \text{if } a < 0. \end{cases}$$

Subjets and superjets

Proposition 2. *Let $u : U \to \mathbb{R}$ where $U \subset \mathbb{R}^n$ is open. We have*

$$J^{2,+}u(x_0) = \left\{ (D\varphi(x_0), D^2\varphi(x_0)) \,:\, \varphi \in C^2(\mathbb{R}^n) \text{ and } u - \varphi \text{ has a local max at } x_0 \right\}.$$

and

$$J^{2,-}u(x_0) = \left\{ (D\varphi(x_0), D^2\varphi(x_0)) \,:\, \varphi \in C^2(\mathbb{R}^n) \text{ and } u - \varphi \text{ has a local min at } x_0 \right\}.$$

Subjets and superjets

We consider the general second order equation

$$F(D^2u, Du, u, x) = 0 \quad \text{in } U. \tag{4}$$

Theorem 2. *Let $U \subset \mathbb{R}^n$ be open and assume F is continuous in all variables. If $u \in USC(\overline{U})$ is a viscosity subsolution of (4) then*

$$F(X, p, u(x), x) \leq 0 \quad \textit{for all } x \in U \textit{ and } (p, X) \in J^{2,+}u(x).$$

Similarly, if $v \in LSC(\overline{U})$ is a viscosity supersolution (4) then

$$F(X, p, v(x), x) \geq 0 \quad \textit{for all } x \in U \textit{ and } (p, X) \in J^{2,-}v(x).$$

Note: This gives an alternative definition of viscosity solutions that is sometimes used in practice.

Consistency

Corollary 1. *Let $U \subset \mathbb{R}^n$ be open and assume F is continuous in all variables. If $u \in USC(\overline{U})$ is a viscosity subsolution of (4) and u is twice differentiable at some $x \in U$ then*

$$F(D^2u(x), Du(x), u(x), x) \leq 0.$$

Similarly, if $v \in LSC(\overline{U})$ is a viscosity supersolution of (4) and v is twice differentiable at some $x \in U$ then

$$F(D^2v(x), Dv(x), v(x), x) \geq 0.$$

Closures of jets

We define

$$\overline{J}_{\mathcal{O}}^{2,+}u(x) = \Big\{(p,X) \in \mathbb{R}^n \times \mathcal{S}(n) \,:\, \exists x_n \in \mathcal{O}, (p_n, X_n) \in J_{\mathcal{O}}^{2,+}u(x_n) \text{ such that } (x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X).\Big\}$$

and

$$\overline{J}_{\mathcal{O}}^{2,-}u(x) = \Big\{(p,X) \in \mathbb{R}^n \times \mathcal{S}(n) \,:\, \exists x_n \in \mathcal{O}, (p_n, X_n) \in J_{\mathcal{O}}^{2,-}u(x_n) \text{ such that } (x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X).\Big\}$$

Theorem on sums

Theorem 3. *Let $\mathcal{O} \subset \mathbb{R}^n$ be locally compact. Let $u \in USC(\mathcal{O})$, $v \in LSC(\mathcal{O})$, and let $\varphi \in C^2(\mathbb{R}^n \times \mathbb{R}^n)$. Suppose $(x_0, y_0) \in \mathcal{O} \times \mathcal{O}$ is a local maximum of*

$$u(x) - v(y) - \varphi(x, y)$$

relative to $\mathcal{O} \times \mathcal{O}$. Then for each κ with $\kappa D^2\varphi(x_0, y_0) < I$ there exists $X, Y \in \mathcal{S}(n)$ such that

$$(D_x\varphi(x_0, y_0), X) \in \overline{J}^{2,+}_{\mathcal{O}} u(x_0), \quad (-D_y\varphi(x_0, y_0), Y) \in \overline{J}^{2,-}_{\mathcal{O}} v(y_0),$$

and the block diagonal matrix with entries $X, -Y$ satisfies

$$-\frac{1}{\kappa} I \leq \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq (I - \kappa D^2\varphi(x_0, y_0))^{-1} D^2\varphi(x_0, y_0).$$

Semi-continuous comparison

We assume $U \subset \mathbb{R}^n$ is open and bounded, F is continuous, degenerate elliptic, and monotone, and satisfies

$$F(X, p, z, y) - F(X, p, z, x) \leq \omega(|x - y|(1 + |p|)) \tag{5}$$

for all $x, y \in U$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$, and $X \in \mathcal{S}(n)$, where ω is a modulus of continuity.

Theorem 4. *Let $u \in USC(\overline{U})$ be a viscosity subsolution of (1), and let $v \in LSC(\overline{U})$ be a viscosity solution of*

$$F(D^2 v, Dv, v, x) - a \geq 0 \quad \text{in } U,$$

for some $a > 0$. If $u \leq v$ on ∂U then $u \leq v$ in U.

Some preparations

Proposition 3. *Let* $\varepsilon > 0$, $u \in USC(\mathbb{R}^n)$, *and* $x_0 \in \mathbb{R}^n$. *Let* $\varphi \in C^\infty(\mathbb{R}^n)$ *such that* $u^\varepsilon - \varphi$ *has a local max at* x_0, *and let* $x_\varepsilon \in \mathbb{R}^n$ *such that*

$$u^\varepsilon(x_0) = u(x_\varepsilon) - \frac{1}{2\varepsilon}|x_0 - x_\varepsilon|^2. \tag{6}$$

Then $u - \psi$ *has a local max at* x_ε *and*

$$D\psi(x_\varepsilon) = D\varphi(x_0) = \frac{1}{\varepsilon}(x_\varepsilon - x_0). \tag{7}$$

Some preparations

Exercise 1. Define
$$w(x,y) = \frac{\alpha}{2}|x-y|^2.$$
Show that the sup-convolution
$$w^\varepsilon(x,y) = \sup_{(x',y')\in\mathbb{R}^n\times\mathbb{R}^n} \left\{ \frac{\alpha}{2}|x'-y'|^2 - \frac{1}{2\varepsilon}|x-x'|^2 - \frac{1}{2\varepsilon}|y-y'|^2 \right\}$$
is given by
$$w^\varepsilon(x,y) = (1-2\alpha\varepsilon)^{-1}\frac{\alpha}{2}|x-y|^2,$$
provided $1-2\alpha\varepsilon \neq 0$.