A PDE-proof of the continuum limit of non-dominated sorting

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Outline

1 Background
   - Motivating example
   - Non-dominated sorting

2 Continuum limit of non-dominated sorting
   - Main Result
   - Non-rigorous derivation
   - Original variational proof

3 PDE proof
   - Monotonicity
   - Stability
   - Consistency
   - Proof

4 Current work
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Motivating example: Google Goggles

Query image

Retrieved images
Facial recognition

**Problem:** Retrieve images containing faces from a large database $S$. 

One approach:

1. Look for features that are characteristic of faces. (Eyes, nose, mouth, ears, etc.)
2. For each of $d$ features, define an objective function $f_i(I) = 1 - \text{Probability that image } I \text{ has feature } i$.
3. Solve the multi-objective optimization problem: 
   $$\text{arg min}_{I \in S} (f_1(I), \ldots, f_d(I)).$$
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Multi-objective optimization

How do we solve the multi-objective optimization problem

$$\arg \min_{I \in S} (f_1(I), \ldots, f_d(I))?$$

Basic approach:
1. Choose some weights $\alpha_i \in [0, 1]$ with $\sum_{i=1}^d \alpha_i = 1$ and define $f_\alpha(I) = \alpha_1 f_1(I) + \alpha_2 f_2(I) + \cdots + \alpha_d f_d(I)$.

2. Solve the scalarized optimization problem $\arg \min_{I \in S} f_\alpha(I)$.

Problems:
1. Difficult to choose weights
2. Ignores relevant solutions
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Non-dominated solutions
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Non-dominated sorting

Let $X_1, \ldots, X_n$ be points in $\mathbb{R}^d$ and set $S = \{X_1, \ldots, X_n\}$.

Define the partial order

$$x \leq y \iff x_i \leq y_i \text{ for all } i \in \{1, \ldots, d\}.$$
Non-dominated sorting

Let $X_1, \ldots, X_n$ be points in $\mathbb{R}^d$ and set $S = \{X_1, \ldots, X_n\}$.

Define the partial order

$$x \preceq y \iff x_i \leq y_i \text{ for all } i \in \{1, \ldots, d\}.$$

**Definition**

Non-dominated sorting is the process of arranging $S$ into layers $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots$, defined by

$$\mathcal{F}_1 = \text{Minimal elements of } S,$$

$$\mathcal{F}_k = \text{Minimal elements of } S \setminus \bigcup_{j \leq k-1} \mathcal{F}_j.$$
Applications

Multi-objective optimization

- Genetic algorithms [Deb et al., 2002]
- Gene selection and ranking [Hero, 2003]
- Database systems [Papadias et al., 2005]
- Anomaly detection [Hsiao et al., 2012]
- Image retrieval [Hsiao et al., 2014]
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**Combinatorics and probability**
- Longest chain in Euclidean space [Hammersley, 1972]
- Patience sorting [Aldous and Diaconis, 1999]
- Young Tableaux [Viennot, 1984]
- Graph theory [Lou and Sarrafzadeh, 1993]
- Polynuclear growth (crystals) [Prähofer and Spohn, 2000]
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**Other applications**
- Molecular biology [Pevzner, 2000]
- Integrated circuit design [Adhar, 2007]
Demo: 50 Random samples
Demo: Uniform distribution
Demo: Uniform distribution
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Demo: Uniform distribution
Demo: Uniform distribution
Demo: Gaussian distribution
Demo: Gaussian distribution
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Question

Can we characterize the asymptotic shapes of the Pareto fronts?
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Assumptions

Let $X_1, \ldots, X_n$ be i.i.d. random variables in $[0, \infty)^d$ with continuous density $f$.

Let $U_n : \mathbb{R}^d \to \mathbb{N}_0$ be the function that ‘counts’ the layers $\mathcal{F}_1, \mathcal{F}_2, \ldots$. 
Main result

**Theorem (Calder, Esedoḡlu, Hero, 2014)**

Let \( X_1, \ldots, X_n \) be i.i.d. with continuous density \( f : [0, \infty)^d \rightarrow [0, \infty) \). Then

\[
n^{-\frac{1}{d}} U_n \longrightarrow u \quad \text{locally uniformly on } [0, \infty)^d
\]

almost surely, where \( u \in C^{0, \frac{1}{d}}([0, \infty)^d) \) is the unique Pareto-monotone \(^ag\) viscosity solution of

\[
(P) \begin{cases}
u_{x_1} \cdots v_{x_d} = \frac{c_d^d}{d^d} f & \text{in } \mathbb{R}_+^d \\ u = 0 & \text{on } \partial \mathbb{R}_+^d.
\end{cases}
\]

\(^ag\) \( x \leq y \implies u(x) \leq u(y) \)
Demo: $f = 1 - \chi_{[0,0.5]^2}$
Demo: Multimodal $f$
Non-rigorous derivation of (P)

Suppose we have $X_1, \ldots, X_n$ i.i.d. with density $f \in C(\mathbb{R}^d)$. 

Non-rigorous derivation of (P)

Suppose we have $X_1, \ldots, X_n$ i.i.d. with density $f \in C(\mathbb{R}^d)$.

Let’s suppose that $n^{-\alpha} U_n \to u \in C^1$ for some $\alpha \in [0, 1]$. Then we should have

$$\mathcal{F}_i \approx \{x \in \mathbb{R}^d : u(x) = in^{-\alpha}\},$$

for $n$ large.
Non-rigorous derivation of (P)

For small $|v|$, \( \langle Du,v \rangle \approx u(x + v) - u(x) \approx n(\text{fronts in } A) - \alpha \approx n(\text{samples in } A) - \alpha \approx \frac{|A|}{f(x)} \alpha n - \alpha \approx |A| \alpha f(x)^\alpha \). Using $|A| \approx \langle Du,v \rangle$ we have $\langle Du,v \rangle \approx (f(x)u_1 \cdots u_d) \alpha \langle Du,v \rangle^\alpha d$.

If $\alpha d = 1$, or $\alpha = 1/d$, then $u_1 \cdots u_d = f(x)$. 

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PDE-proof continuum limit

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Non-rigorous derivation of (P)

For small $|v|$

\[
\ell_1 = \frac{\langle Du, v \rangle}{u_{x_1}}
\]

\[
\ell_2 = \frac{\langle Du, v \rangle}{u_{x_2}}
\]

\[
\langle Du, v \rangle \approx u(x + v) - u(x)
\]
Non-rigorous derivation of (P)

For small $|v|$

\[
\langle Du, v \rangle \approx u(x + v) - u(x) \\
\approx (\# \text{ fronts in } A)n^{-\alpha}
\]
Non-rigorous derivation of (P)

For small $|v|$

$$\langle Du, v \rangle \approx u(x + v) - u(x)$$
$$\approx \left( \# \text{ fronts in } A \right) n^{-\alpha}$$
$$\approx \left( \# \text{ samples in } A \right)^\alpha n^{-\alpha}$$
Non-rigorous derivation of (P)

For small $|v|$

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\langle Du, v \rangle \approx u(x + v) - u(x)
\]

\[
\approx \left( \# \text{ fronts in } A \right) n^{-\alpha}
\]

\[
\approx \left( \# \text{ samples in } A \right) \alpha n^{-\alpha}
\]

\[
\approx \left( n |A| f(x) \right) \alpha n^{-\alpha}
\]
Non-rigorous derivation of (P)

For small $|v|$

\[ \langle Du, v \rangle \approx u(x + v) - u(x) \]
\[ \approx (\# \text{ fronts in } A) n^{-\alpha} \]
\[ \approx (\# \text{ samples in } A)^{\alpha} n^{-\alpha} \]
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Non-rigorous derivation of (P)

For small $|v|

\langle Du, v \rangle \approx u(x + v) - u(x)
\approx (\# \text{ fronts in } A)n^{-\alpha}
\approx (\# \text{ samples in } A)\alpha n^{-\alpha}
\approx (n|A|f(x))^\alpha n^{-\alpha}
\approx |A|^\alpha f(x)^\alpha.

Using $|A| \approx \frac{\langle Du, v \rangle^d}{u_{x1} \cdots u_{xd}}$ we have

\langle Du, v \rangle \approx \left( \frac{f(x)}{u_{x1} \cdots u_{xd}} \right)^\alpha \langle Du, v \rangle^\alpha d
Non-rigorous derivation of (P)

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$$\langle Du, v \rangle \approx u(x + v) - u(x) \approx (\# \text{ fronts in } A)n^{-\alpha} \approx (\# \text{ samples in } A)^{\alpha}n^{-\alpha} \approx (n|A|f(x))^{\alpha}n^{-\alpha} \approx |A|^\alpha f(x)^\alpha.$$

Using $|A| \approx \frac{\langle Du, v \rangle^d}{u_{x_1} \cdots u_{x_d}}$ we have

$$\langle Du, v \rangle \approx \left( \frac{f(x)}{u_{x_1} \cdots u_{x_d}} \right)^\alpha \langle Du, v \rangle^\alpha^d$$

If $\alpha d = 1$, or $\alpha = 1/d$, then

$$u_{x_1} \cdots u_{x_d} = f$$
Non-rigorous derivation of (P)

Seems difficult to make this argument rigorous for two reasons

1. Convergence $n^{-\frac{1}{d}} U_n \rightarrow u$ not obvious
2. Requires $u \in C^1$
Original proof

Our original proof [Calder et al., 2014] was based on the continuum variational problem

\[
u(x) = c_d \cdot \sup \left\{ \int_0^1 f(\gamma(t))^{\frac{1}{d}} (\gamma'_1(t) \cdots \gamma'_d(t))^{\frac{1}{d}} dt : \gamma'(t) \in \mathbb{R}^d_+ \text{ and } \gamma(1) = x \right\}. \tag{1}
\]
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\[ u(x) = c_d \cdot \sup \left\{ \int_0^1 f(\gamma(t))^{\frac{1}{d}} (\gamma_1'(t) \cdots \gamma_d'(t))^{\frac{1}{d}} \, dt : \gamma'(t) \in \mathbb{R}_+^d \text{ and } \gamma(1) = x \right\}. \quad (1) \]

The Hamilton-Jacobi equation

\[
(P) \begin{cases} 
  u_{x_1} \cdots u_{x_d} = \frac{c_d^d}{d^d} f & \text{in } \mathbb{R}_+^d \\
  u = 0 & \text{on } \partial \mathbb{R}_+^d,
\end{cases}
\]

is the Hamilton-Jacobi-Bellman equation for this variational problem (1).

The variational problem (1) appeared originally in [Deuschel and Zeitouni, 1995] in dimension \( d = 2 \).
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Towards a PDE-proof

We will give a new proof using only PDE techniques.

Main Ideas:
- View non-dominated sorting as a numerical scheme for (P).
- (Roughly) follow the [Barles and Souganidis, 1991] framework for convergence of numerical schemes to viscosity solutions.

Monotonicity + Consistency + Stability + Well-posedness $\implies$ Convergence.
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Main Ideas:
- View non-dominated sorting as a numerical scheme for (P).
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  Monotonicity + Consistency + Stability + Well-posedness $\implies$ Convergence.

Question: What do monotonicity, consistency, and stability refer to in this stochastic setting?
Key observation

\[ \mathcal{F}_1 \quad \mathcal{F}_2 \quad \mathcal{F}_3 \quad \mathcal{F}_4 \quad \mathcal{F}_5 \quad \mathcal{F}_6 \]
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Alternative definition of $U_n$

A chain in $S$ is a sequence $x_1, \ldots, x_\ell$ such that

$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_\ell$$

For $S \subseteq \mathbb{R}^d$ let

$$\ell(S) = \text{Length of a longest chain in } S.$$
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For $S \subseteq \mathbb{R}^d$ let

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We can alternatively define $U_n : \mathbb{R}^d \rightarrow \mathbb{N}_0$ by

$$U_n(x) := \ell \left( \{ X_i : X_i \leq x \} \right).$$

![Diagram showing the function $U_n(x)$ with different values for $U_n$ at various points in $\mathbb{R}^d$.](image)
Poisson point process

For convenience, we will model the data as a Poisson point process.

Given a locally integrable non-negative function $f : \mathbb{R}^d \to \mathbb{R}$, we denote by $\Pi_f$ the associated Poisson point process with rate function $f$. This means:

1. $\Pi_f$ is a random countable subset of $\mathbb{R}^d$.
2. For every $A \subseteq \mathbb{R}^d$, $N(A) := \#(\Pi_f \cap A)$ is a Poisson random variable with mean $\int_A f$.
3. For disjoint sets $A, B \subseteq \mathbb{R}^d$, $N(A)$ and $N(B)$ are independent.

Suppose $\int_{\mathbb{R}^d} f = 1$. Let $X_1, X_2, X_3, \ldots$ i.i.d. with density $f$ and let $N$ be a Poisson random variable with mean $n$. Then $\Pi_{nf} = \{X_1, \ldots, X_N\}$.

We define $U_n(x) = \ell(\Pi_{nf} \cap [0, x])$, where $[0, x] = [0, x_1] \times \cdots \times [0, x_d]$. 
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\[
\Pi_{nf} = \{X_1, \ldots, X_N\}.
\]

We define

\[
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\]

where \([0, x] = [0, x_1] \times \cdots \times [0, x_d] \).
Monotonicity

If $A \subseteq B$ then

$$\ell(\Pi_{nf} \cap A) \leq \ell(\Pi_{nf} \cap B).$$
Theorem ([Hammersley, 1972])

There exists a constant $c_d$ such that

$$\ell \left( \Pi_n \cap [0,1]^d \right) \sim c_d n^{\frac{1}{d}} \text{ almost surely.}$$
Longest chain in a cube

**Theorem ([Hammersley, 1972])**

There exists a constant $c_d$ such that

$$\ell \left( \prod_n \cap [0,1]^d \right) \sim c_d n^{1/d} \quad \text{almost surely.}$$

Longest chain problem has a long history in probability and combinatorics

- Ulam’s famous problem [Ulam, 1961]
- [Bollobás and Winkler, 1988]

$$\frac{d^2}{d! \sqrt{\frac{1}{d}} \Gamma \left( \frac{1}{d} \right)} \leq c_d < e \quad \text{for all } d \geq 1.$$

- [Deuschel and Zeitouni, 1995], [Aldous and Diaconis, 1995]
Some simple observations

1. For a rectangle $A = \prod_{i=1}^{d} [a_i, b_i]$, a scaling argument gives

$$\ell(\Pi_n \cap A) \sim c_d |A|^\frac{1}{d} n^\frac{1}{d}$$

almost surely,

where $|A| = (b_1 - a_1) \cdots (b_d - a_d)$. 

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   where $|A| = (b_1 - a_1) \cdots (b_d - a_d)$.

2. For bounded $f : \mathbb{R}^d \to [0, \infty)$

   \[ \ell(\Pi_n f \cap A) \leq \ell(\Pi_n \|f\|_{L^\infty} \cap A), \]

   which gives

   \[ \ell(\Pi_n f \cap A) \lesssim c_d |A|^{\frac{1}{d}} \|f\|_{L^\infty}^{\frac{1}{d}} n^{\frac{1}{d}} \text{ almost surely.} \]

   \[ X_n \lesssim Cn^{\frac{1}{d}} \iff \limsup_{n \to \infty} n^{-\frac{1}{d}} X_n \leq C. \]
\[ U_n(x) = \ell \left( \Pi_{nf} \cap [0, x] \right) \]
Stability

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\[ U_n(x) = \ell(\Pi_{nf} \cap [0, x]) = \ell(c_1) + \ell(c_2) \leq U_n(y) + \ell(\Pi_{nf} \cap A). \]
Stability

For every \( x, y \in \mathbb{R}^d \)

\[
U_n(x) - U_n(y) \lesssim C(R) \|f\|_{L^\infty} \frac{1}{n} |x - y|^\frac{1}{d} n^\frac{1}{d},
\]

almost surely, where \( R = \max\{x_1, y_1, \ldots, x_d, y_d\} \).
Stability

For every $x, y \in \mathbb{R}^d$

$$U_n(x) - U_n(y) \lesssim C(R)\|f\|_{L^\infty} \frac{1}{d^n} |x - y|^{\frac{1}{d}} n^{\frac{1}{d}},$$

almost surely, where $R = \max\{x_1, y_1, \ldots, x_d, y_d\}$.

Using the monotonicity of $U_n$ ($U_n(x + he_i) \geq U_n(x)$ for $h > 0$), this can be improved to

**Theorem (Stability)**

$$P \left( \forall x, y \in \mathbb{R}^d, U_n(x) - U_n(y) \lesssim C(R)\|f\|_{L^\infty} \frac{1}{d^n} |x - y|^{\frac{1}{d}} n^{\frac{1}{d}} \right) = 1.$$
Longest chain in a simplex

\[ S = \{ x \in [0, 1]^d : x_1 + \cdots + x_d \geq d - 1 \} \]

\[ \ell(\Pi_n \cap S) \sim \]

\[ \ell(\Pi_n \cap S) \sim c_d d^n \text{ almost surely} \]
Longest chain in a simplex

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\[ \ell(\Pi_n \cap S) \sim \ell \left( \Pi_n \cap [1 - 1/d, 1]^d \right) \]
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\[ \ell(\Pi_n \cap S) \sim \ell \left( \Pi_n \cap [1 - 1/d, 1]^d \right) \sim \frac{c_d}{d} n^{\frac{1}{d}} \text{ almost surely.} \]
Longest chain in a simplex

For a simplex $S$ with side-lengths $v_1, \ldots, v_d$, a scaling argument yields

\[ \ell(\Pi_n \cap S) \sim \frac{c_d}{d} (v_1 \cdots v_d)^{\frac{1}{d}} n^{\frac{1}{d}} \text{ a.s.} \]
Longest chain in a simplex

For a simplex $S$ with side-lengths $v_1, \ldots, v_d$, a scaling argument yields

$$\ell(\Pi_n \cap S) \sim \frac{c_d}{d} (v_1 \cdots v_d)^{\frac{1}{d}} n^{\frac{1}{d}} \text{ a.s.}$$

**Lemma**

For bounded $f: \mathbb{R}^d \to \mathbb{R}$, we have

$$\frac{c_d}{d} (v_1 \cdots v_d)^{\frac{1}{d}} \left( \inf_{S} f \right)^{\frac{1}{d}} n^{\frac{1}{d}} \lesssim \ell(\Pi_{nf} \cap S) \lesssim \frac{c_d}{d} (v_1 \cdots v_d)^{\frac{1}{d}} \left( \sup_{S} f \right)^{\frac{1}{d}} n^{\frac{1}{d}},$$

almost surely.
Consistency

Let \( x \in [0, \infty)^d \) and \( \varphi \in C^2(\mathbb{R}^d) \) with \( \varphi_{x_i}(x) > 0 \) for all \( i \). Let \( \varepsilon > 0 \) and set

\[
S_\varepsilon = \left\{ y \in [0, x] : \varphi(y) \geq \varphi(x) - \varepsilon \right\}.
\]

\[
\{ y \in \mathbb{R}^d : \varphi(y) = \varphi(x) \}
\]

\[
\approx \varphi_{x_1}^{-1} \varepsilon
\]

\[
\approx \varphi_{x_2}^{-1} \varepsilon
\]

\[
\{ y \in \mathbb{R}^d : \varphi(y) = \varphi(x) - \varepsilon \}
\]

\[
D \varphi(x)
\]

\[
x
\]
Recall

\[ S_\varepsilon = \left\{ y \in [0, x] : \varphi(y) \geq \varphi(x) - \varepsilon \right\}. \]

Applying the lemma, we have the following consistency statement

**Theorem (Consistency)**

*Suppose \( f \) is continuous. Then*

\[
P \left( \forall x \in \mathbb{R}^d \ \forall \varphi \in X, \ \ell \left( \prod_{nf} \cap S_\varepsilon \right) \sim \frac{c_d}{d} \left( \frac{f(x)}{\varphi_{x_1}(x) \cdots \varphi_{x_d}(x)} \right)^{\frac{1}{d}} \left( \varepsilon + O(\varepsilon^2) \right) n^{\frac{1}{d}} \right) = 1,
\]

*where*

\[ X = \left\{ \varphi \in C^2(\mathbb{R}^d) : \varphi_{x_i} > 0 \right\}. \]
Proof

Recall

\[ U_n(x) = \ell(\Pi_n \cap [0, x]). \]

Stability + Consistency

There exists a probability one event \( \Omega \) such that \( U_n^\omega \equiv 0 \) on \( \partial \mathbb{R}_+^d \),

\[ U_n^\omega(x) - U_n^\omega(y) \lesssim C(R)\|f\|_{L^\infty}^{1/d} |x - y|^{1/d} n^{1/d}, \]

and

\[ \ell(\Pi_n^\omega \cap S_\varepsilon) \sim \frac{c_d}{d} \left( \frac{f(x)}{\varphi_{x_1}(x) \cdots \varphi_{x_d}(x)} \right)^{1/d} (\varepsilon + O(\varepsilon^2)) n^{1/d} \]

for all \( \omega \in \Omega \), all \( x, y \in \mathbb{R}_+^d \), and all \( \varphi \in C^2(\mathbb{R}_+^d) \) with \( \varphi_{x_i} > 0 \) for all \( i \), where

\[ S_\varepsilon = \left\{ y \in [0, x] : \varphi(y) \geq \varphi(x) - \varepsilon \right\}. \]
Proof

**Stability:**

\[ U_n^\omega(x) - U_n^\omega(y) \lesssim C(R) \| f \|_{L^\infty} \frac{1}{n^{\frac{d}{d}}} |x - y|^{\frac{1}{d}} n^{\frac{1}{d}} \]

for all \( \omega \in \Omega \) and \( x, y \in \mathbb{R}^d \).
Proof

**Stability:**

\[ U_n^\omega(x) - U_n^\omega(y) \lesssim C(R) \| f \|_{L^\infty} \frac{1}{n} |x - y| \frac{1}{d} n^\frac{1}{d} \]

for all \( \omega \in \Omega \) and \( x, y \in \mathbb{R}^d \).

Using an argument based on Arzelà-Ascoli, for every \( \omega \in \Omega \) there exists a subsequence \( U_{n_k}^\omega \) and a function \( U^\omega \in C^{\frac{1}{d}}(\mathbb{R}^d) \) such that

\[ n_k^{-\frac{1}{d}} U_{n_k}^\omega \rightarrow U^\omega \quad \text{locally uniformly on } \mathbb{R}^d, \]

and \( U^\omega \equiv 0 \) on \( \partial \mathbb{R}^d_+ \).
Proof

**Stability:**

\[ U_n^\omega(x) - U_n^\omega(y) \lesssim C(R) \| f \|_{L^\infty} \frac{1}{n^d} |x - y|^{\frac{1}{d}} \]

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and \( U^\omega \equiv 0 \) on \( \partial \mathbb{R}^d_+ \).

We will show that \( U^\omega \) is a viscosity solution of (P) for every \( \omega \in \Omega \).
Recall: Viscosity solution

Consider the Hamilton-Jacobi equation

\[ H(Du) = f \quad \text{on } \mathcal{O}, \quad (2) \]

where \( \mathcal{O} \subseteq \mathbb{R}^d \) is open, and \( H : \mathbb{R}^d \rightarrow \mathbb{R} \) and \( f : \mathcal{O} \rightarrow \mathbb{R} \) are continuous.

A continuous function \( U : \mathcal{O} \rightarrow \mathbb{R} \) is a viscosity solution of (2) if

1. **Subsolution:** For every \( x \in \mathcal{O} \) and \( \varphi \in C^\infty(\mathcal{O}) \) such that \( U - \varphi \) has a local maximum at \( x \)

   \[ H(D\varphi(x)) \leq f(x). \]

2. **Supersolution:** For every \( x \in \mathcal{O} \) and \( \varphi \in C^\infty(\mathcal{O}) \) such that \( U - \varphi \) has a local minimum at \( x \)

   \[ H(D\varphi(x)) \geq f(x). \]
Proof

$U^\omega$ subsolution of (P):

Fix $\omega \in \Omega$. Let $x \in (0, R)^d$ and $\varphi \in C^2(\mathbb{R}^d)$ such that $U^\omega - \varphi$ has a strict maximum at $x$ relative to $[0, R]^d$. Then there exists $x_n \to x$ such that $n^{-\frac{1}{d}} U_n^\omega - \varphi$ has a maximum at $x_n$ relative to $[0, R]^d$. 

Since $x_n \to x$ and $n^{-\frac{1}{d}} U_n^\omega \to U^\omega$ locally uniformly, for large enough $n$ we have $n^{-\frac{1}{d}} U_n^\omega(y) - n^{-\frac{1}{d}} U_n^\omega(x) \leq \varphi(y) - \varphi(x) + \epsilon$. Therefore $S_n,\epsilon \subseteq \{y \in [0, x] : \varphi(y) \geq \varphi(x) - \epsilon - \epsilon/2\} =: S_\epsilon$. 

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Proof

**$U^\omega$ subsolution of (P):**
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$$S_{n, \varepsilon} = \left\{ y \in [0, x] : n^{-\frac{1}{d}} U_n^\omega (y) \geq n^{-\frac{1}{d}} U_n^\omega (x) - \varepsilon \right\}.$$
Proof

$U^\omega$ subsolution of (P):

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Since

$$n^{-\frac{1}{d}} U^\omega_n(y) - \varphi(y) \leq n^{-\frac{1}{d}} U^\omega_n(x_n) - \varphi(x_n) \quad \text{for all } y \in [0, R]^d,$$

We have

$$n^{-\frac{1}{d}} U^\omega_n(y) - n^{-\frac{1}{d}} U^\omega_n(x) \leq \varphi(y) - \varphi(x) + \varphi(x) - \varphi(x_n) + n^{-\frac{1}{d}} U^\omega_n(x_n) - n^{-\frac{1}{d}} U^\omega_n(x).$$
Proof

$U^\omega$ subsolution of (P):
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We have

$$n^{-\frac{1}{d}} U_n^\omega(y) - n^{-\frac{1}{d}} U_n^\omega(x) \leq \varphi(y) - \varphi(x) + \varphi(x) - \varphi(x_n) + n^{-\frac{1}{d}} U_n^\omega(x_n) - n^{-\frac{1}{d}} U_n^\omega(x).$$

Since $x_n \to x$ and $n^{-\frac{1}{d}} U_n^\omega \to U^\omega$ locally uniformly, for large enough $n$ we have

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$$n^{-\frac{1}{d}} U_n^\omega(y) - n^{-\frac{1}{d}} U_n^\omega(x) \leq \varphi(y) - \varphi(x) + \varepsilon^2.$$

Therefore

$$S_{n,\varepsilon} \subseteq \{ y \in [0, x] : \varphi(y) \geq \varphi(x) - \varepsilon - \varepsilon^2 \} =: S_\varepsilon.$$
Proof

Recall

\[ S_{n,\varepsilon} = \left\{ y \in [0, x] : n^{-\frac{1}{d}} U_n^\omega (y) \geq n^{-\frac{1}{d}} U_n^\omega (x) - \varepsilon \right\}. \]

By monotonicity

\[ \ell \left( \Pi_{nf}^\omega \cap \left\{ y \in [0, x] : \varphi (y) \geq \varphi (x) - \varepsilon - \varepsilon^2 \right\} \right) \geq \ell (\Pi_{nf}^\omega \cap S_{n,\varepsilon}) \]
Proof

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Proof

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By consistency

\[ \varepsilon \leq \limsup_{n \to \infty} n^{-\frac{1}{d}} \ell \left( \Pi_{nf}^\omega \cap \left\{ y \in [0, x] : \varphi(y) \geq \varphi(x) - \varepsilon - \varepsilon^2 \right\} \right) \leq \frac{c_d}{d} \left( \frac{f(x)}{\varphi_{x_1}(x) \cdots \varphi_{x_d}(x)} \right)^{\frac{1}{d}} (\varepsilon + O(\varepsilon^2)). \]
Proof

Recall

\[ S_{n, \varepsilon} = \left\{ y \in [0, x] : n^{-\frac{1}{d}} U_n^\omega(y) \geq n^{-\frac{1}{d}} U_n^\omega(x) - \varepsilon \right\}. \]

By monotonicity

\[
\ell \left( \Pi_{n_f}^\omega \cap \{ y \in [0, x] : \varphi(y) \geq \varphi(x) - \varepsilon - \varepsilon^2 \} \right) \geq \ell(\Pi_{n_f}^\omega \cap S_{n, \varepsilon}) \geq \varepsilon n^{\frac{1}{d}}. 
\]

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\[
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\leq \frac{c_d}{d} \left( \frac{f(x)}{\varphi_{x_1}(x) \cdots \varphi_{x_d}(x)} \right)^{\frac{1}{d}} (\varepsilon + O(\varepsilon^2)).
\]

Sending \( \varepsilon \to 0^+ \) we find that

\[
\varphi_{x_1}(x) \cdots \varphi_{x_d}(x) \leq \frac{c_d}{d^d} f(x).
\]
Proof

\( U^\omega \) a supersolution of (P): Similar proof, but an additional lemma is needed:

**Lemma**

There exists a, possibly smaller, probability one event \( \Omega \) such that for all \( \omega \in \Omega \), all \( x \in [0, \infty)^d \) such that \( f(x) > 0 \), and all \( \varphi \in C^2(\mathbb{R}^d) \) such that \( U^\omega - \varphi \) has a strict minimum at \( x \), we have

\[
\varphi_{x_i}(x) > 0 \quad \text{for all } i.
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Proof

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**Lemma**

There exists a, possibly smaller, probability one event $\Omega$ such that for all $\omega \in \Omega$, all $x \in [0, \infty)^d$ such that $f(x) > 0$, and all $\varphi \in C^2(\mathbb{R}^d)$ such that $U^\omega - \varphi$ has a strict minimum at $x$, we have

$$\varphi_x(x) > 0 \quad \text{for all } i.$$

Therefore $U^\omega = u$ for all $\omega \in \Omega$, where $u$ is the unique viscosity solution of

\[
\begin{aligned}
(P) \quad \left\{ 
\begin{array}{ll}
    u_{x_1} \cdots u_{x_d} = \frac{c_d}{d!} f & \quad \text{in } \mathbb{R}^d_+ \\
    u = 0 & \quad \text{on } \partial \mathbb{R}^d_+.
\end{array}
\right.
\end{aligned}
\]

This completes the proof. $\square$
Outline

1 Background
   - Motivating example
   - Non-dominated sorting

2 Continuum limit of non-dominated sorting
   - Main Result
   - Non-rigorous derivation
   - Original variational proof

3 PDE proof
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   - Stability
   - Consistency
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4 Current work
   - Convex hull peeling

5 References
The ordering of multivariate data is an important and challenging problem.
Ordering of multivariate data

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- No obvious candidate for concepts like ‘median’ in dimensions $d \geq 2$. 

Barnett [Barnett, 1976] introduced the idea of convex hull ordering

▶ Idea is to sort points in Euclidean space into layers by repeatedly removing the vertices of the convex hull.
▶ Also known as convex hull peeling or onion peeling (or ‘the onion’).
▶ Convex hull peeling median is the ‘center’ of the onion

Many different types of orderings and definitions of median exist in the literature

▶ [Barnett, 1976], [Small, 1990], [Liu et al., 1999]
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Convex hull peeling

Let $X_1, \ldots, X_n$ be points in $\mathbb{R}^d$ and set $S = \{X_1, \ldots, X_n\}$.

Recall: Convex hull of a set $\mathcal{O}$ is the intersection of all convex sets containing $\mathcal{O}$. 
Convex hull peeling

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Recall: Convex hull of a set $O$ is the intersection of all convex sets containing $O$.

**Definition**

Convex hull peeling is the process of arranging $S$ into convex layers $C_1, C_2, C_3, \ldots$, defined by

$$
C_1 = \text{Vertices of convex hull of } S,
$$
$$
C_k = \text{Vertices of convex hull of } S \setminus \bigcup_{j \leq k-1} C_j.
$$
Convex hull peeling

Figure: Depiction of convex layers for (a) \( n = 10^2 \) and (b) \( n = 10^4 \) independent and uniformly distributed points \( X_1, \ldots, X_n \) on \([0, 1]^2\).
Applications

Convex hull peeling is widely used in robust statistics, machine learning, etc.

- [Rousseeuw and Struyf, 2004], [Donoho and Gasko, 1992], [Hodge and Austin, 2004].

Matching of deformed pointclouds [Suk and Flusser, 1999].

Convex layers invariant under affine transformations.

Fingerprint identification [Poulos et al., 2005].

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Let $X_1, \ldots, X_n$ be i.i.d. on a convex open and bounded set $\Omega \subseteq \mathbb{R}^2$ with density $f : \Omega \to [0, \infty)$. Let $C_1, C_2, \ldots$ denote the associated convex layers. Define

$$U_n(x) = \sup \left\{ k \in \mathbb{N} : x \in \text{ConvHull}(C_k) \right\}.$$
Conjectured continuum limit

Let $X_1, \ldots, X_n$ be i.i.d. on a convex open and bounded set $\Omega \subseteq \mathbb{R}^2$ with density $f : \Omega \rightarrow [0, \infty)$. Let $C_1, C_2, \ldots$ denote the associated convex layers. Define

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According to [Dalal, 2004], there are in expectation $O\left( n^{\frac{2}{3}} \right)$ convex layers.
If we assume that $n^{-\frac{2}{3}} U_n \to u$, some basic heuristic scaling arguments can be made that suggest

$$n^{-\frac{2}{3}} U_n \to u \quad \text{uniformly on } \overline{\Omega}$$

almost surely, where $u$ is the viscosity solution of

$$|Du| \left( \text{div} \left( \frac{Du}{|Du|} \right) \right)^{\frac{1}{3}} + cf(x, y)^{\frac{2}{3}} = 0, \quad \text{in } \Omega, \quad U = 0, \quad \text{on } \partial \Omega. \quad \text{(3)}$$

This is Affine Invariant Curvature Motion.
Conjectured continuum limit

Figure: Visual comparison of convex layers and affine invariant curvature motion for $n = 5 \times 10^3$ i.i.d. samples from (a) a circular and (b) a triangular domain $\Omega$. 
Figure: (a) $L^1$ and (b) $L^\infty$ errors between the solution $u$ of the affine invariant curvature motion PDE (3) and the convex depth function $U_n$, both normalized to range from 0 to 1. The errors appear to be $O(n^{-\frac{1}{2}})$ in all test cases.
Thanks!
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5. References


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Counting the onion.

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