Numerical Schemes for the Hamilton-Jacobi Equation
Continuum Limit of Non-dominated Sorting

Jeff Calder

Department of Mathematics
University of California, Berkeley

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McGill University

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Outline

1 Introduction
   - Motivating example: Image retrieval
   - Non-dominated sorting

2 Continuum limit for nondominated sorting
   - Hamilton-Jacobi equation for layers
   - PDE-based ranking

3 Numerical schemes
   - An $O(h^{1/n})$ scheme
   - Two (formally) $O(h)$ schemes
   - Regularity
   - Convergence rates

4 Experimental results

5 References
Motivating example: Google Goggles

Query image

Retrieved images
Multi-query image retrieval

**Problem:** Find images in a dataset $S$ that are similar to multiple query images.

**Pareto method:** Solve the multi-criteria optimization problem

$$\arg\min_{I \in S} (\text{dist}(I, Q_1), \ldots, \text{dist}(I, Q_d)).$$

Query 1

Query 2
Multi-query image retrieval

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**Pareto method:** Solve the multi-criteria optimization problem

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**Pareto points:**

[Diagram showing Pareto points]
Multi-objective optimization

How do we solve the multi-objective optimization problem

$$\arg\min_{I \in S} (f_1(I), \ldots, f_d(I))$$?
Multi-objective optimization

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$$\arg \min_{I \in S} (f_1(I), \ldots, f_d(I))?$$

Basic approach:

1. Choose some weights $\alpha_i \in [0, 1]$ with $\sum_{i=1}^{d} \alpha_i = 1$ and define

$$f_\alpha(I) = \alpha_1 f_1(I) + \alpha_2 f_2(I) + \cdots + \alpha_d f_d(I).$$
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2. Solve the scalarized optimization problem

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2. Solve the scalarized optimization problem

   $$\arg \min_{I \in S} f_\alpha(I).$$

Problems:

1. Difficult to choose weights
2. Ignores relevant solutions
Basic approach
Basic approach
Basic approach
Non-dominated solutions
Non-dominated solutions
Non-dominated solutions
Non-dominated solutions
Non-dominated solutions
Non-dominated solutions
Non-dominated solutions
Multi-query image retrieval

First Pareto front:

Query 1
Query 2
1  2  3  4  5
6  7  8  9  10
11 12 13 14 15

Non-dominated sorting

Let $X_1, \ldots, X_N$ be points in $\mathbb{R}^n$ and set $S = \{X_1, \ldots, X_N\}$.

Define the partial order

$$x \leq y \iff x_i \leq y_i \text{ for all } i \in \{1, \ldots, n\}.$$
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\[ x \leq y \iff x_i \leq y_i \text{ for all } i \in \{1, \ldots, n\}. \]

**Definition**

Non-dominated sorting is the process of arranging $S$ into layers $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots$, defined by

\[ \mathcal{F}_1 = \text{Minimal elements of } S, \]

\[ \mathcal{F}_k = \text{Minimal elements of } S \setminus \bigcup_{j \leq k-1} \mathcal{F}_j. \]
Applications

Multi-objective optimization

- Genetic algorithms [Deb et al., 2002]
- Gene selection and ranking [Hero, 2003]
- Database systems [Papadias et al., 2005]
- Anomaly detection [Hsiao et al., 2012, Hsiao et al., 2015b]
- Image retrieval [Hsiao et al., 2015a]
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Combinatorics and probability
- Longest chain in Euclidean space [Hammersley, 1972]
- Patience sorting [Aldous and Diaconis, 1999]
- Young Tableaux [Viennot, 1984]
- Graph theory [Lou and Sarrafzadeh, 1993]
- Polynuclear growth (crystals) [Prähofer and Spohn, 2000]
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Other applications
- Molecular biology [Pevzner, 2000]
- Integrated circuit design [Adhar, 2007]
Demo: 50 Random samples
Demo: Uniform distribution

\[ N = 10^2 \text{ points} \]
Demo: Uniform distribution

\[ N = 10^3 \text{ points} \]
Demo: Uniform distribution

\[ N = 10^4 \text{ points} \]
Demo: Uniform distribution

\[ N = 10^5 \text{ points} \]
Demo: Uniform distribution

\( N = 10^6 \) points
Demo: Gaussian distribution

\[ N = 10^2 \text{ points} \]
Demo: Gaussian distribution

\[ N = 10^3 \text{ points} \]
Demo: Gaussian distribution

$N = 10^4$ points
Demo: Gaussian distribution

\[ N = 10^5 \text{ points} \]
Demo: Gaussian distribution

\[ N = 10^6 \text{ points} \]
Demo: Uniform distribution on $[0, 1]^2 \setminus [0, 0.5]^2$

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Demo: Uniform distribution on $[0, 1]^2 \setminus [0, 0.5]^2$

$N = 10^6$ points
Question

Can we characterize the asymptotic shapes of the Pareto fronts?
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A PDE continuum limit for non-dominated sorting

Let $X_1, \ldots, X_N$ be i.i.d. random variables in $[0, \infty)^n$ with continuous density $f$.

Let $U_N : \mathbb{R}^n \rightarrow \mathbb{N}_0$ be the function that ‘counts’ the layers $\mathcal{F}_1, \mathcal{F}_2, \ldots$
Continuum limit

**Theorem (Calder, Esedoḡlu, Hero, 2014)**

*With probability one*

\[
N^{-\frac{1}{d}} U_N \longrightarrow u \quad \text{locally uniformly on } [0, \infty)^n,
\]

where \( u \in C^{0,\frac{1}{n}} ([0, \infty)^n) \) is the unique nondecreasing viscosity solution of

\[
(P1) \begin{cases}
    u_{x_1} \cdots u_{x_n} = c_n f & \text{in } \mathbb{R}^n_+ := (0, \infty)^n \\
    u = 0 & \text{on } \partial \mathbb{R}^n_+.
\end{cases}
\]


Demo: $f = 1 - \chi_{[0,0.5]^2}$
Demo: Multimodal $f$
Demo: A Cat
Fast approximate sorting

Algorithm (PDE-based Ranking)

1. Select \( k \) points from \( X_1, \ldots, X_N \) at random. Call them \( Y_1, \ldots, Y_k \).
2. Select a grid spacing \( h \) for solving the PDE \((P1)\) and estimate \( f \) with a histogram aligned to the grid \([0, 1]^n_h\), i.e.,

\[
\hat{f}(x) = \frac{1}{kh^n} \cdot \# \left\{ Y_i : x \leq Y_i \leq x + h(1, \ldots, 1) \right\} \quad \text{for } x \in [0, 1]^n_h.
\]
3. Compute the numerical solution \( \hat{U}_h \) on \([0, 1]^n_h\).
4. Evaluate \( \hat{U}_h(X_i) \) for \( i = 1, \ldots, N \) via interpolation.

Notes:

- Total complexity is \( O(k + h^{-n} + N) \).
- If we fix \( k, h \) and \( n \), independent of \( N \), then Steps 1-3 have \( O(1) \) complexity.

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How do we numerically solve

\begin{align*}
\begin{split}
\{ & \frac{\partial u}{\partial x_1} \cdots \frac{\partial u}{\partial x_n} = f & \quad \text{in } \mathbb{R}^n_+ \\
& u = 0 & \quad \text{on } \partial \mathbb{R}^n_+ 
\end{split}
\end{align*}

efficiently and accurately (in dimensions $n = 2, 3, 4$)?
Recall: Viscosity solution

Consider the Hamilton-Jacobi equation

$$H(x, u, Du) = 0 \text{ on } \mathcal{O} \subseteq \mathbb{R}^n,$$  \hspace{1cm} (1)

where $H : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

A continuous function $u : \mathcal{O} \to \mathbb{R}$ is a viscosity solution of (1) if

1. **Subsolution:** For every $x \in \mathcal{O}$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at $x$ with respect to $\mathcal{O}$

$$H(x, u(x), D\varphi(x)) \leq 0.$$  

2. **Supersolution:** For every $x \in \mathcal{O}$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at $x$ with respect to $\mathcal{O}$

$$H(x, u(x), D\varphi(x)) \geq 0.$$
Bounded domain

We first pose the PDE on a bounded domain

\[
(P1) \begin{cases} 
    u_{x_1} \cdots u_{x_n} = f & \text{in } (0, 1]^n \\
    u = 0 & \text{on } \Gamma 
\end{cases}
\]

where \( \Gamma = [0, 1]^n \setminus (0, 1]^n \).

Minor technicality: Viscosity solutions of (P1) do not exist due to a well-known issue with viscosity solutions on boundaries of domains. We actually need to slightly modify the PDE:

\[
(P1') \begin{cases} 
    (u_{x_1}) + \cdots + (u_{x_n}) = f & \text{in } (0, 1]^n \\
    u = 0 & \text{on } \Gamma 
\end{cases}
\]

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We actually need to slightly modify the PDE:

\[
(P1') \begin{cases} 
  (u_{x1})_+ \cdots (u_{xn})_+ = f \quad \text{in } (0, 1]^n \\
  u = 0 \quad \text{on } \Gamma,
\end{cases}
\]

where \( t_+ = \max(t, 0) \).
Upwind finite difference scheme for \((P1)\)

The upwind scheme corresponds to using backward differences:

\[
(S1) \begin{cases}
D_1^- u_h(x) \cdots D_n^- u_h(x) = f(x) & \text{for } x \in (0, 1]^n \\
u_h(x) = 0 & \text{for } x \in \Gamma_h,
\end{cases}
\]

where \(\Omega_h := \Omega \cap h\mathbb{Z}^n\) and

\[
D_i^\pm u_h(x) = \pm \frac{u_h(x \pm he_i) - u_h(x)}{h}.
\]
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D_i^\pm u_h(x) = \pm \frac{u_h(x \pm he_i) - u_h(x)}{h}.
\]

The scheme \((S)\) can be solved in a single pass (similar to fast sweeping or marching), and in dimension \(n = 2\) we have the closed form expression

\[
u_h(x) = \frac{u_h(x - he_1) + u_h(x - he_2)}{2} + \frac{1}{2} \sqrt{(u_h(x - he_1) - u_h(x - he_2))^2 + 4h^2 f(x)^2}.
\]

However, accuracy is poor \(O(h^{\frac{1}{n}})\).

Accuracy of (S1)

For $f \equiv 1$, $u(x) = n(x_1 \cdots x_n)^{\frac{1}{n}}$. Set $\varphi(x) = Cn(x_1 \cdots x_n)^{\frac{1}{n}}$ and by concavity

$$D_i^− \varphi(x) \geq \varphi_{x_i}(x) = C(x_1 \cdots x_n)^{\frac{1}{n}} x_i^{-1}.$$
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$$D_i^- \varphi(x) \geq \varphi_{x_i}(x) = C(x_1 \cdots x_n)^{\frac{1}{n}} x_i^{-1}.$$  

On the other hand, if $x_i = h$ then

$$D_i^- \varphi(x) = \frac{\varphi(x)}{h} = Cn(x_1 \cdots x_n)^{\frac{1}{n}} x_i^{-1}.$$
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D_i^- \varphi(x) = \frac{\varphi(x)}{h} = Cn(x_1 \cdots x_n)^{\frac{1}{n}} x_i^{-1}.
\]

Therefore, for any \( x \in (0, 1]^n_h \) such that \( x_i = h \) for some \( i \) we have

\[
D_1^- \varphi(x) \cdots D_n^- \varphi(x) \geq nC^n := 1 \quad \left( C := \frac{1}{n^{\frac{1}{n}}} \right).
\]
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Therefore, for any $x \in (0, 1]_h^n$ such that $x_i = h$ for some $i$ we have

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By comparison, $u_h(x) \leq \varphi(x)$ whenever $x_i = h$ for some $i$. 
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\]

By comparison, \( u_h(x) \leq \varphi(x) \) whenever \( x_i = h \) for some \( i \). For \( x = (h, 1, \ldots, 1) \)

\[
u_h(x) \leq \varphi(x) = n^{1-\frac{1}{n}} h^{\frac{1}{n}} \quad \text{and} \quad u(x) = nh^{\frac{1}{n}}.
\]
Accuracy of (S1)

For \( f \equiv 1 \), \( u(x) = n(x_1 \cdots x_n)^{\frac{1}{n}} \). Set \( \varphi(x) = C n(x_1 \cdots x_n)^{\frac{1}{n}} \) and by concavity

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D_i^- \varphi(x) \geq \varphi_{x_i}(x) = C (x_1 \cdots x_n)^{\frac{1}{n}} x_i^{-1}.
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On the other hand, if \( x_i = h \) then

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D_i^- \varphi(x) = \frac{\varphi(x)}{h} = C n(x_1 \cdots x_n)^{\frac{1}{n}} x_i^{-1}.
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Therefore, for any \( x \in (0, 1]^n_h \) such that \( x_i = h \) for some \( i \) we have

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D_1^- \varphi(x) \cdots D_n^- \varphi(x) \geq n C^n := 1 \quad \left( C := \frac{1}{n^{\frac{1}{n}}} \right).
\]

By comparison, \( u_h(x) \leq \varphi(x) \) whenever \( x_i = h \) for some \( i \). For \( x = (h, 1, \ldots, 1) \)

\[
u_h(x) \leq \varphi(x) = n^{1 - \frac{1}{n}} h^{\frac{1}{n}} \quad \text{and} \quad u(x) = n h^{\frac{1}{n}}.
\]

Therefore

\[
u(x) - u_h(x) \geq \left( n - n^{1 - \frac{1}{n}} \right) h^{\frac{1}{n}}.
\]
Towards a new scheme

This suggests that we should try to remove the gradient singularity. Consider

\[ v := \frac{u^n}{n^n}. \]
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\[ v := \frac{u^n}{n^n}. \]

Then

\[ v_{x_i} = \frac{u^{n-1}}{n^{n-1}} u_{x_i} = v^n u_{x_i}. \]

Therefore

\[ v_{x_1} \cdots v_{x_n} = v^{n-1} u_{x_1} \cdots u_{x_n} = v^{n-1} f. \]
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Therefore

\[ v_{x_1} \cdots v_{x_n} = v^{n-1} u_{x_1} \cdots u_{x_n} = v^{n-1} f. \]

We find that \( v \) is a viscosity solution of

\[
(P2) \begin{cases}
  v_{x_1} \cdots v_{x_n} = v^{n-1} f & \text{in } (0, 1]^n \\
  v = 0 & \text{on } \Gamma
\end{cases}
\]
Towards a new scheme

This suggests that we should try to remove the gradient singularity. Consider

\[ v := \frac{u^n}{n^n}. \]

Then

\[ v_x = \frac{n}{n-1} u_x = v \frac{n-1}{n} u_x. \]

Therefore

\[ v_1 \cdots v_n = v^{n-1} u_1 \cdots u_n = v^{n-1} f. \]

We find that \( v \) is a viscosity solution of

\[ (P2) \begin{cases} v_1 \cdots v_n = v^{n-1} f & \text{in } (0,1)^n \\ v = 0 & \text{on } \Gamma \end{cases} \]

Since \( f \geq 0 \), (P2) has a zeroth order term of the wrong sign for a comparison principle. The method of vanishing viscosity takes the form

\[ v_1^\varepsilon \cdots v_n^\varepsilon - \varepsilon \Delta v^\varepsilon = (v^\varepsilon)^{n-1} f. \]
Nonuniqueness for (P2)

\[
(P2) \begin{cases} 
  v_{x_1} \cdots v_{x_n} = v^{n-1} f & \text{in } (0, 1]^n \\
  v = 0 & \text{on } \Gamma
\end{cases}
\]

When \( f \equiv 1 \), \( v(x) = u(x)^n / n^n = x_1 \cdots x_n \), but

\[ v^y(x) := (x_1 - y_1)_+ \cdots (x_n - y_n)_+ \]

is also a viscosity solution of (P2) for any \( y \in (0, 1)^n \).
Nonuniqueness for (P2)

\[
\begin{aligned}
(P2) \quad \begin{cases}
v_{x_1} \cdots v_{x_n} = v^{n-1}f & \text{in } (0,1]^n \\
v = 0 & \text{on } \Gamma
\end{cases}
\end{aligned}
\]

When \( f \equiv 1 \), \( v(x) = u(x)^n/n^n = x_1 \cdots x_n \), but

\[
v^y(x) := (x_1 - y_1)^+ \cdots (x_n - y_n)^+
\]

is also a viscosity solution of (P2) for any \( y \in (0,1)^n \).

Lemma

Assume \( f \) is continuous on \([0,1]^n\). Then \( v := u^n/n^n \) is the unique maximal viscosity solution of (P2).
Numerical scheme for (P2)

An upwind scheme for (P2) uses backward difference quotients

\[(S2) \begin{cases} D_1^- v_h(x) \cdots D_n^- v_h(x) = v_{h, 1}^{n-1}(x)f(x) \quad \text{for } x \in (0, 1]^n_h \\ v_h(x) = 0 \quad \text{for } x \in \Gamma_h, \end{cases}\]

We define $v_h$ by taking the largest solution of (S2) at each $x \in (0, 1]^n_h$. 

Notice when $f \equiv 1$, $v_h(x) = x_1 \cdots x_n$ is the exact solution of (P2).
Numerical scheme for (P2)

An upwind scheme for (P2) uses backward difference quotients

\[
(S2) \begin{cases} 
    D_1^- v_h(x) \cdots D_n^- v_h(x) = v_h^{n-1}(x)f(x) & \text{for } x \in (0, 1]_h \\
    v_h(x) = 0 & \text{for } x \in \Gamma_h,
\end{cases}
\]

We define \( v_h \) by taking the largest solution of \((S2)\) at each \( x \in (0, 1]_h \).

The scheme can be solved efficiently in a single pass, and in dimension \( n = 2 \) the scheme can be solved in closed form

\[
v_h(x) = \frac{A + h^2 f(x)}{2} + \frac{1}{2} \sqrt{B^2 + 2h^2 f(x)A + h^4 f(x)^2},
\]

where

\[
A = v_h(x - he_1) + v_h(x - he_2) \quad \text{and} \quad B = v_h(x - he_1) - v_h(x - he_2).
\]
Numerical scheme for (P2)

An upwind scheme for (P2) uses backward difference quotients

(S2) \[
\begin{align*}
D_1^- v_h(x) \cdots D_n^- v_h(x) &= v_h^{n-1}(x)f(x) \quad \text{for } x \in (0, 1]^n_h \\
v_h(x) &= 0 \quad \text{for } x \in \Gamma_h,
\end{align*}
\]

We define $v_h$ by taking the largest solution of (S2) at each $x \in (0, 1]^n_h$.

The scheme can be solved efficiently in a single pass, and in dimension $n = 2$ the scheme can be solved in closed form

\[
v_h(x) = \frac{A + h^2f(x)}{2} + \frac{1}{2} \sqrt{B^2 + 2h^2f(x)A + h^4f(x)^2},
\]

where

\[
A = v_h(x - he_1) + v_h(x - he_2) \quad \text{and} \quad B = v_h(x - he_1) - v_h(x - he_2).
\]

Notice when $f \equiv 1$, $v_h(x) = x_1 \cdots x_n$ is the exact solution of (P2).
Rate of convergence for (S2)

\[
(S2) \begin{cases}
  D_{1}^{-} v_h(x) \cdots D_{n}^{-} v_h(x) = v_h^{n-1}(x)f(x) & \text{for } x \in (0, 1)_h^n \\
v_h(x) = 0 & \text{for } x \in \Gamma_h,
\end{cases}
\]

The scheme (S2) has formal accuracy of $O(h)$ and we have

**Theorem**

Suppose $f \in C^{0,1}([0, 1]^n)$ and $f > 0$. Then

\[
|n^n v_h^n - u^n| \leq C \sqrt{h} \quad \text{in } [0, 1]_h^n, \tag{2}
\]

and

\[
|nv_{h}^{1/n} - u| \leq C \delta^{1-n} \sqrt{h} \quad \text{in } [\delta, 1]_h^n, \tag{3}
\]

where $\delta > 0$ and $C = C(n, [f]_{1;[0,1]^n}, \min_{[0,1]^n} f)$.

One-sided rate for (S1)

Recall $u_h$ satisfies

$$D_1^- u_h(x) \cdots D_n^- u_h(x) = f(x) \quad \text{for } x \in (0, 1]^n.$$
One-sided rate for (S1)

Recall $u_h$ satisfies

$$D_1^- u_h(x) \cdots D_n^- u_h(x) = f(x) \quad \text{for } x \in (0, 1]^n.$$ 

Let $\psi(x) = u_h^n/n^n$. By convexity and monotoncity

$$D_i^- \psi(x) = \frac{u_h(x)^n - u_h(x - he_i)^n}{n^n h} \leq \frac{u_h(x)^{n-1}}{n^{n-1}} D_i^- u_h(x).$$
One-sided rate for (S1)

Recall \( u_h \) satisfies

\[
D_1^- u_h(x) \cdots D_n^- u_h(x) = f(x) \quad \text{for } x \in (0, 1]_h.
\]

Let \( \psi(x) = \frac{u_h}{n^n} \). By convexity and monotonicity

\[
D_i^- \psi(x) = \frac{u_h(x)^n - u_h(x - he_i)^n}{n^n h} \leq \frac{u_h(x)^{n-1}}{n^{n-1}} D_i^- u_h(x).
\]

Therefore

\[
D_1^- \psi(x) \cdots D_n^- \psi(x) \leq \psi(x)^{n-1} f(x),
\]

so \( \psi \) is a subsolution of (S2). By comparison we have \( u_h / n^n = \psi \leq v_h \).
One-sided rate for (S1)

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Therefore

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D_1^- \psi(x) \cdots D_n^- \psi(x) \leq \psi(x)^{n-1} f(x),
\]

so \( \psi \) is a subsolution of (S2). By comparison we have \( u_h^n / n^n = \psi \leq v_h \).

Corollary

Suppose \( f \in C^{0,1}([0, 1]^n) \) and \( f > 0 \). Then

\[
u_h^n - u^n \leq C \sqrt{h}.
\]
Regularity

To prove the $O(\sqrt{h})$ convergence rate, we need a Lipschitz regularity result for (P2) or (S2). Textbook regularity results are based on ($p = Du$)

\[(\text{Coercivity}) \quad \lim_{|p| \to \infty} H(x, p) = \infty \quad \text{uniformly in} \ x,\]

or

\[(\text{Zeroth order term}) \quad u + H(x, Du) = 0.\]

Refer to [Bardi and Dolcetta, 1997]
Regularities

To prove the $O(\sqrt{h})$ convergence rate, we need a Lipschitz regularity result for (P2) or (S2). Textbook regularity results are based on ($p = Du$)

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or

\[(\text{Zeroth order term}) \quad u + H(x, Du) = 0.\]

Refer to [Bardi and Dolcetta, 1997]

The Hamiltonian for (P2) is ($z = u, p = Du$)

\[H(x, z, p) = p_1 \cdots p_n - z^{n-1} f(x)\]

which satisfies neither.
Regularity for $n = 2$

Differentiate both sides of (P2) $\frac{\partial}{\partial x}(v_x v_y) = \frac{\partial}{\partial x}(vf)$ to find that
\[ v_{xx} v_y + v_x v_{xy} = v_x f + vf_x. \]
Regularity for $n = 2$

Differentiate both sides of (P2) $\frac{\partial}{\partial x}(v_x v_y) = \frac{\partial}{\partial x}(vf)$ to find that

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Set $\varphi = v_x$ and rearrange

$$\varphi_x v_y + (\varphi_y - f)\varphi = vf_x \leq \|f\|_{L^\infty[1;[0,1]^n}.$$

We can compare $\varphi$ against a supersolution of the form $\varphi(x,y) = C(1 + y)$.

The bound $v(x) \leq \|f\|_{L^\infty}$ yields boundary gradient estimates.

Lemma

Let $f \in C^0_0([0,1]^2)$ be nonnegative. Then there exists $C > 0$ such that

$$\|v\|_{1;[0,1]^2} \leq C \|f\|_{C^0_0([0,1]^2)}.$$ (4)
Regularity for $n = 2$

Differentiate both sides of (P2) $\frac{\partial}{\partial x}(v_x v_y) = \frac{\partial}{\partial x}(vf)$ to find that

$$v_{xx} v_y + v_x v_{xy} = v_x f + vf_x.$$ 

Set $\varphi = v_x$ and rearrange

$$\varphi_x v_y + (\varphi_y - f)\varphi = vf_x \leq \|f\|_{L^\infty} [f]_{1;[0,1]^n}.$$ 

We can compare $\varphi$ against a supersolution of the form

$$\overline{\varphi}(x, y) = C(1 + y).$$

The bound $v(x) \leq \|f\|_{L^\infty} xy$ yields boundary gradient estimates.
Regularity for \( n = 2 \)

Differentiate both sides of (P2) \( \frac{\partial}{\partial x} (v_x v_y) = \frac{\partial}{\partial x} (vf) \) to find that

\[
v_{xx} v_y + v_x v_{xy} = v_x f + vf_x.
\]

Set \( \varphi = v_x \) and rearrange

\[
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We can compare \( \varphi \) against a supersolution of the form

\[
\overline{\varphi}(x, y) = C(1 + y).
\]

The bound \( v(x) \leq \|f\|_{L^\infty} xy \) yields boundary gradient estimates.

**Lemma**

Let \( f \in C^{0,1}([0,1]^2) \) be nonnegative. Then there exists \( C' > 0 \) such that

\[
[v]_{1;[0,1]^2} \leq C' \|f\|_{C^{0,1}([0,1]^2)}.
\]
Sketch of proof

Differentiate numerical scheme $D_x^-(D_x^- v D_y^- v) = D_x^-(vf)$ \hspace{1cm} (x = (x, y))

\[ D_x^- D_x^- v(x) D_y^- v(x - he_x) + D_x^- v(x) D_x^- D_y^- v(x) = v(x) D_x^- f(x) + f(x - he_x) D_x^- v(x). \]
Sketch of proof

Differentiate numerical scheme $D_x^- (D_x^- v D_y^- v) = D_x^- (vf)$ \hspace{1cm} $(x = (x, y))$

\[ D_x^- D_x^- v(x) D_y^- v(x - he_x) + D_x^- v(x) D_x^- D_y^- v(x) = v(x) D_x^- f(x) + f(x - he_x) D_x^- v(x). \]

Set $\varphi = D_x^- v$ to find

\[
D_x^- \varphi(x) D_y^- v(x - he_x) + \left( D_y^- \varphi(x) - f(x - he_x) \right) \varphi(x) = vD_x^- f \leq \|f\|_{L^\infty} [f].
\]
Sketch of proof

Differentiate numerical scheme \( D_x^- (D_x^- v D_y^- v) = D_x^- (vf) \) \( (x = (x, y)) \)

\[
D_x^- D_x^- v(x) D_y^- v(x - he_x) + D_x^- v(x) D_x^- D_y^- v(x) = v(x) D_x^- f(x) + f(x - he_x) D_x^- v(x).
\]

Set \( \varphi = D_x^- v \) to find

\[
D_x^- \varphi(x) D_y^- v(x - he_x) + \left( D_y^- \varphi(x) - f(x - he_x) \right) \varphi(x) = vD_x^- f \leq \|f\|_{L^\infty} [f]_1.
\]

Set \( \overline{\varphi}(x, y) = C(1 + y) \) and compute

\[
\left( D_y^- \overline{\varphi}(x) - f(x - he_x) \right) \overline{\varphi}(x) \geq C \left( C - \|f\|_{L^\infty} \right).
\]
Sketch of proof

Differentiate numerical scheme $D_x^- (D_x^- v D_y^- v) = D_x^- (v f) \quad (x = (x, y))$

$D_x^- D_x^- v(x) D_y^- v(x - h e_x) + D_x^- v(x) D_x^- D_y^- v(x) = v(x) D_x^- f(x) + f(x - h e_x) D_x^- v(x)$.

Set $\varphi = D_x^- v$ to find

$$D_x^- \varphi(x) D_y^- v(x - h e_x) + \left( D_y^- \varphi(x) - f(x - h e_x) \right) \varphi(x) = v D_x^- f \leq \|f\|_{L^\infty} [f]_1.$$

Set $\bar{\varphi}(x, y) = C'(1 + y)$ and compute

$$\left( D_y^- \varphi(x) - f(x - h e_x) \right) \bar{\varphi}(x) \geq C \left( C - \|f\|_{L^\infty} \right).$$

We can select

$$C' = \|f\|_{L^\infty} + \sqrt{\|f\|_{L^\infty} [f]_1}$$

and we have

$$\left( D_y^- \varphi(x) - f(x - h e_x) \right) \bar{\varphi}(x) > \|f\|_{L^\infty} [f]_1.$$
Sketch of proof

We claim $\varphi \leq \bar{\varphi} := C(1 + y)$. 

Since $v(x, y) \leq \|f\|_{L^\infty}$, we have $\varphi \leq \varphi$ on $\Gamma = [0, 1]^2$.

Assume to the contrary that $\varphi(x) > \varphi(x)$ for some $x \in (0, 1]^2$. Then there exists $x$ such that $\varphi(x) > \varphi(x)$, $\varphi(x - h) \leq \varphi(x - h)$, and $\varphi(x - h) \leq \varphi(x - h)$.

Then $D - x \varphi(x) \geq D - x \varphi(x) = 0$ and $D - y \varphi(x) \geq D - y \varphi(x) \geq \|f\|_{L^\infty}$.

Hence $\left(D - y \varphi(x) - f(x - h)\right) \varphi(x) \geq \left(D - y \varphi(x) - f(x - h)\right) \varphi(x) \geq \|f\|_{L^\infty}$.

Recalling $D - x \varphi(x) D - y v(x - h) \geq 0 + \left(D - y \varphi(x) - f(x - h)\right) \varphi(x) = vD - x f \leq \|f\|_{L^\infty}$, we have a contradiction.

Thus $D - x v \leq 2C = 2\left(\|f\|_{L^\infty} + \sqrt{\|f\|_{L^\infty} [f]}^1\right) \leq C\|f\|_{C^0, 1}$.
Sketch of proof

We claim \( \varphi \leq \varphi := C(1 + y) \). Since \( v(x, y) \leq \| f \|_{L^\infty xy} \), \( \varphi \leq \varphi \) on \( \Gamma = [0, 1]^2 \setminus (0, 1]^2 \).
Sketch of proof

We claim $\varphi \leq \bar{\varphi} := C(1 + y)$. Since $v(x, y) \leq \|f\|_{L^\infty} xy$, $\varphi \leq \bar{\varphi}$ on $\Gamma = [0, 1]^2 \setminus (0, 1]^2$. Assume to the contrary that $\varphi(x) > \bar{\varphi}(x)$ for some $x \in (0, 1]^2$. Then there exists $x$ such that

$$
\varphi(x) > \bar{\varphi}(x), \quad \varphi(x - he_x) \leq \bar{\varphi}(x - he_x), \quad \text{and} \quad \varphi(x - he_y) \leq \bar{\varphi}(x - he_y).
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Then

$$D^-_x \varphi(x) \geq D^-_x \overline{\varphi}(x) = 0 \quad \text{and} \quad D^-_y \varphi(x) \geq D^-_y \overline{\varphi}(x) \geq \|f\|_{L^\infty}. $$
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Then

$$D_x^- \varphi(x) \geq D_x^- \overline{\varphi}(x) = 0 \quad \text{and} \quad D_y^- \varphi(x) \geq D_y^- \overline{\varphi}(x) \geq \|f\|_{L^\infty}.$$

Hence

$$
(D_y^- \varphi(x) - f(x - he_x))\varphi(x) \geq (D_y^- \overline{\varphi}(x) - f(x - he_x))\overline{\varphi}(x).
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Hence

$$(D_y^- \varphi(x) - f(x - he_x))\varphi(x) \geq (D_y^- \overline{\varphi}(x) - f(x - he_x))\overline{\varphi}(x).$$

Recalling

$$D_x^- \varphi(x) D_y^- v(x - he_x) + \left( D_y^- \varphi(x) - f(x - he_x) \right) \varphi(x) = vD_x^- f \leq \|f\|_{L^\infty} [f]_1.$$

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Then

$$D_x^- \varphi(x) \geq D_x^- \overline{\varphi}(x) = 0 \quad \text{and} \quad D_y^- \varphi(x) \geq D_y^- \overline{\varphi}(x) \geq \|f\|_{L^\infty}.$$

Hence

$$\left( D_y^- \varphi(x) - f(x - he_x) \right) \varphi(x) \geq \left( D_y^- \overline{\varphi}(x) - f(x - he_x) \right) \overline{\varphi}(x).$$

Recalling

$$\underbrace{D_x^- \varphi(x) D_y^- v(x - he_x)}_{\geq 0} + \left( D_y^- \varphi(x) - f(x - he_x) \right) \varphi(x) = vD_x^- f \leq \|f\|_{L^\infty} [f]_1.$$

we have a contradiction. Thus

$$D_x^- v \leq 2C = 2 \left( \|f\|_{L^\infty} + \sqrt{\|f\|_{L^\infty} [f]_1} \right) \leq C \|f\|_{C^{0, 1}}. \quad \square$$
Regularity for \( n = 3 \)

Differentiate both sides of (P2) \( \frac{\partial}{\partial x}(v_x v_y v_z) = \frac{\partial}{\partial x}(v^2 f) \) to find that

\[
v_{xx} v_y v_z + v_x v_{yx} v_z + v_x v_y v_{zx} = 2vv_x f + v^2 f_x.
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Set $\varphi = v_x$ and rearrange

$$F(\varphi) := \varphi_x v_y v_z + (v_z \varphi_y + v_y \varphi_z - 2vf)\varphi = v^2 f_x \leq \|f\|^2_{L^\infty} [f]_1.$$
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Set $\varphi = v_x$ and rearrange

$$F(\varphi) := \varphi_x v_y v_z + (v_z \varphi_y + v_y \varphi_z - 2vf)\varphi = v^2 f_x \leq \|f\|_{L^\infty}^2 [f].$$

Perhaps we should be looking for a supersolution of the form

$$\overline{\varphi}(x, y, z) = C(1 + y + z)?$$

Then $F(\overline{\varphi}) \geq C(Cv_z + Cv_y - 2vf)$...
Regularity for $n = 3$

Differentiate both sides of (P2) $\frac{\partial}{\partial x} (v_x v_y v_z) = \frac{\partial}{\partial x} (v^2 f)$ to find that

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Perhaps we should be looking for a supersolution of the form

$$\bar{\varphi}(x, y, z) = C(1 + y + z)?$$

Then $F(\bar{\varphi}) \geq C(Cv_z + Cv_y - 2vf) \ldots$

At a maximum of $\varphi$, $\varphi_x = \varphi_y = \varphi_z = 0$ so

$$-2vf \varphi \leq \|f\|^2_{L^\infty}[f]_1.$$
Regularity for $n \geq 3$?

Unfortunately this argument fails for $n \geq 3$ due to the additional nonlinearity.

Furthermore, we cannot directly prove rates for (P2) due to the wrong sign on zeroth order term.
Regularity for $n \geq 3$?

Unfortunately this argument fails for $n \geq 3$ due to the additional nonlinearity.

Furthermore, we cannot directly prove rates for (P2) due to the wrong sign on zeroth order term.

Back to the drawing board!
Another PDE

Recall for \( f \equiv 1 \), \( u(x) = n(x_1 \cdots x_n)^{\frac{1}{n}} \). For general \( f \), we are tempted to make the ansatz

\[
u(x) = n(x_1 \cdots x_n)^{\frac{1}{n}} w(x).
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Another PDE

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$$u(x) = n(x_1 \cdots x_n)^{\frac{1}{n}} w(x).$$

Proceeding formally

$$u_{x_i} = (x_1 \cdots x_n)^{\frac{1}{n}} \frac{w}{x_i} + n(x_1 \cdots x_n)^{\frac{1}{n}} w_{x_i} = \frac{1}{x_i} (x_1 \cdots x_n)^{\frac{1}{n}} (w + nx_i w_{x_i}).$$

Therefore

$$f = u_{x_1} \cdots u_{x_n} = \prod_{i=1}^{n} (w + nx_i w_{x_i}).$$
Another PDE

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Proceeding formally

\[
u_{x_i} = (x_1 \cdots x_n)^{1/n} \frac{w}{x_i} + n(x_1 \cdots x_n)^{1/n} w_{x_i} = \frac{1}{x_i} (x_1 \cdots x_n)^{1/n} (w + nx_i w_{x_i}).
\]

Therefore

\[
f = u_{x_1} \cdots u_{x_n} = \prod_{i=1}^{n} (w + nx_i w_{x_i}).
\]

We can show that \( w \) is a bounded viscosity solution of

\[
(P3) \quad \prod_{i=1}^{n} (w + nx_i w_{x_i}) = f \quad \text{in } (0, 1]^n.
\]

Notice:

- Zeroth order term has correct sign! (since \( u_{x_i} \geq 0 \))
- \( v = u^n / n^n = x_1 \cdots x_n w^n \).
Regularity for (P3)

Let us rewrite (P3) as

\[ \sum_{j=1}^{n} \log(w + nx_j w_{x_j}) = \log(f) \quad \text{in } (0, 1]^n. \]
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Let us rewrite (P3) as

\[
\sum_{j=1}^{n} \log(w + nx_j w_{x_j}) = \log(f) \quad \text{in } (0, 1]^n.
\]

Differentiate each side in \( x_i \):

\[
\sum_{j=1}^{n} \frac{w_{x_i} + n\delta_{i,j} w_{x_j} + nx_j w_{x_i x_j}}{w + nx_j w_{x_j}} = \frac{f_{x_i}}{f}.
\]
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\[
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\]

At a maximum of \( w_{x_i} \) we have \( w_{x_i} w_{x_j} = 0 \) hence

\[
w_{x_i} \sum_{j=1}^{n} \frac{1 + n\delta_{i,j}}{w + nx_j w_{x_j}} = \frac{f_{x_i}}{f}.
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Regularity for (P3)
Let us rewrite (P3) as
\[
\sum_{j=1}^{n} \log(w + nx_j w_{x_j}) = \log(f) \quad \text{in } (0, 1]^n.
\]

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\[
\sum_{j=1}^{n} \frac{w_{x_i} + n\delta_{i,j} w_{x_j} + nx_j w_{x_i} w_{x_j}}{w + nx_j w_{x_j}} = \frac{f_{x_i}}{f}.
\]

At a maximum of \(w_{x_i}\) we have \(w_{x_i} x_j = 0\) hence
\[
w_{x_i} \sum_{j=1}^{n} \frac{1 + n\delta_{i,j}}{w + nx_j w_{x_j}} = \frac{f_{x_i}}{f}.
\]

By the inequality of arithmetic and geometric means
\[
\sum_{j=1}^{n} \frac{1 + n\delta_{i,j}}{w + nx_j w_{x_j}} \geq \sum_{j=1}^{n} \frac{1}{w + nx_j w_{x_j}} \geq n \prod_{j=1}^{n} (w + nx_j w_{x_j})^{-\frac{1}{n}} = nf^{-\frac{1}{n}}.
\]
Regularity for \((P3)\)

Hence, at a positive maximum of \(w_{x_i}\) we have

\[
w_{x_i} \leq \frac{1}{n} f^{\frac{1}{n} - 1} f_{x_i} = (f^{\frac{1}{n}})_{x_i} \leq [f^{\frac{1}{n}}]_{1;[0,1]^n}.
\]
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Hence, at a positive maximum of $w_{xi}$ we have

$$w_{xi} \leq \frac{1}{n} f^{\frac{1}{n}-1} f_{xi} = (f_{\frac{1}{n}})_{xi} \leq [f_{\frac{1}{n}}]_{1:[0,1]^n}.$$ 

**Problem:** We have no boundary condition for $w$, how do we know $w_{xi}$ attains a maximum or minimum?
Regularity for (P3)

Hence, at a positive maximum of \( w_{x_i} \) we have

\[
 w_{x_i} \leq \frac{1}{n} f_{\frac{1}{n}}^{n-1} f x_i = (f_{\frac{1}{n}} x_i) \leq [f_{\frac{1}{n}}]_{n;[0,1]}. 
\]

**Problem:** We have no boundary condition for \( w \), how do we know \( w_{x_i} \) attains a maximum or minimum?

Recall

\[
 (P3) \prod_{i=1}^{n} (w + nx_i w_{x_i}) = f \quad \text{in } (0,1)^n. 
\]

When \( f \equiv 1 \), the solution of (P3) that we care about is \( w \equiv 1 \). For any \( C > 0 \)

\[
 w(x) = \prod_{i=1}^{n} (1 + C x_i^{-1})^{\frac{1}{n}} 
\]

is an unbounded solution of (P3).
Extension property for (P3)

\[(P3) \quad \prod_{i=1}^{n}(w + nx_i w_{x_i}) = f \quad \text{in } (0, 1)^n.\]

**Lemma**

Let $f \geq 0$ be continuous on $[0, 1]^n$ and let $u$ be the solution of (P1). Then

\[w(x) = \frac{u(x)}{n(x_1 \cdots x_n)^{\frac{1}{n}}}\]

can be extended to a continuous function $w \in C((-\infty, 1]^n)$ satisfying

\[w(x) = w(x_+) \quad \text{for all } x \in (-\infty, 1]^n,\]

where $x_+ = (\max(x_1, 0), \ldots, \max(x_n, 0))$. Furthermore, if we extend $f$ by setting $f(x) := f(x_+)$ then $w$ is a viscosity solution of

\[\prod_{i=1}^{n}(w + nx_i w_{x_i}) = f \quad \text{in } (-\infty, 1]^n.\]
Regularity for (P3)

The extension property allows us to complete the regularity result.

**Theorem**

Let $f$ be a nonnegative function for which $f^{\frac{1}{n}} \in C^{0,1}([0, 1]^n)$, and let $w$ be the maximal bounded viscosity solution of (P3). Then $w \in C^{0,1}([0, 1]^n)$ and

$$[w]_{1,[0,1]^n} \leq \sqrt{n} \left[f^{\frac{1}{n}}\right]_{1,[0,1]^n}.$$  \hfill (5)

Theorem is sharp: When $f(x) = (x_1 \cdots x_n)^{n-1}$, $w(x) = \frac{1}{n} (x_1 \cdots x_n)^{1-\frac{1}{n}} \not\in C^{0,1}([0, 1]^n)$. 
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Theorem is sharp: When $f(x) = (x_1 \cdots x_n)^{n-1}$, $w(x) = \frac{1}{n} (x_1 \cdots x_n)^{1-\frac{1}{n}} \not\in C^{0,1}([0, 1]^n)$.

Since $v = x_1 \cdots x_n w^n$ we have the following corollary.

**Corollary**

Let $v$ be the maximal viscosity solution of (P2). Then $v \in C^{0,1}([0, 1]^n)$ and

$$[v]_{1;[0,1]^n} \leq C \|f^{\frac{n-1}{n}}\|_{L^\infty([0,1]^n)} \|f^{\frac{1}{n}}\|_{C^{0,1}([0,1]^n)}. \quad (6)$$
Numerical scheme for (P3)

An upwind scheme for (P3) uses backward difference quotients

\[(S3) \quad \prod_{i=1}^{n} (w(x) + nx_i D^-_i w(x)) = f(x) \quad \text{for } x \in [0, 1]^n.\]

We define \(w_h\) by taking the largest solution of (P2) at each \(x \in [0, 1]^n\).

Notice

- \(w_h(0) = f(0)^{\frac{1}{n}}\).
Numerical scheme for (P3)

An upwind scheme for (P3) uses backward difference quotients

\[ (S3) \prod_{i=1}^{n} (w_h(x) + nx_i D^-_i w_h(x)) = f(x) \quad \text{for } x \in [0, 1]^n_h. \]

We define \( w_h \) by taking the largest solution of (P2) at each \( x \in [0, 1]^n_h \).

**Notice**

- \( w_h(0) = f(0) \frac{1}{n} \).
- When \( f \equiv 1 \), \( w_h \equiv 1 \) which is the exact solution of (P3).
Numerical scheme for (P3)

An upwind scheme for (P3) uses backward difference quotients

\[
(S3) \quad \prod_{i=1}^{n} \left( w_h(x) + nx_i D_i^- w_h(x) \right) = f(x) \quad \text{for } x \in [0, 1]^n_h.
\]

We define \( w_h \) by taking the largest solution of (P2) at each \( x \in [0, 1]^n_h \).

Notice

- \( w_h(0) = f(0)^{\frac{1}{n}}. \)
- When \( f \equiv 1 \), \( w_h \equiv 1 \) which is the exact solution of (P3).

The scheme can be solved efficiently in a single pass, and in dimension \( n = 2 \) the scheme can be solved in closed form

\[
w_h(x) = C + \sqrt{D^2 + (2x_1 + h)(2x_2 + h)h^2 f(x)},
\]

where

\[
C = x_1(2x_2 + h)w_h(x - he_1) + x_2(2x_1 + h)w_h(x - he_2),
\]

and

\[
D = x_1(2x_2 + h)w_h(x - he_1) - x_2(2x_1 + h)w_h(x - he_2).
\]
Convergence rate

Since (P3) has a zeroth order term of the correct sign, and \( w \) and \( w_h \) are Lipschitz continuous on \([0, 1]^n\) we can prove

**Theorem**

Suppose that \( f \in C^{0,1}([0, 1]^n) \) and \( f > 0 \). Then

\[
|w(x) - w_h(x)| \leq C \sqrt{h} \quad \text{for all } x \in [0, 1]_h^n,
\]

(7)

where \( C = C(n, [f]_1;[0,1]^n, \min_{[0,1]^n} f) \).
Directed Hamiltonians

All of the PDE in this talk have a common useful property.

\[ H(x, u(x), Du(x)) = 0. \]

**Definition**

We say that \( H(x, z, p) \) is **directed** if

\[ H(x, z, p) \leq H(x, z, q) \text{ whenever } p_i \leq q_i \text{ for all } i. \]
Directed Hamiltonians

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\[ H(x, z, p) \leq H(x, z, q) \quad \text{whenever} \quad p_i \leq q_i \quad \text{for all} \quad i. \]

Recall:

(P1) \[ H(x, z, p) = (p_1)_+ \cdots (p_n)_+ - f(x) \]
(P2) \[ H(x, z, p) = (p_1)_+ \cdots (p_n)_+ - z^{n-1} f(x) \]
(P3) \[ H(x, z, p) = \prod_{i=1}^{n} (z + nx_i p_i)_+ - f(x) \]
Monotone (upwind) scheme for directed $H$

A monotone scheme for a directed Hamiltonian $H$ is always to use backward differences

$$H(x, u(x), D^- u(x)) = 0,$$

where

$$D^- u(x) = (D_1^- u(x), \ldots, D_n^- u(x)).$$
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where

$$D^- u(x) = (D_1^- u(x), \ldots, D_n^- u(x)).$$

Indeed, if $u \geq v$ and $u(x) = v(x)$ then $D_i^- u(x) \leq D_i^- v(x)$ for all $i$ and

$$H(x, u(x), D^- u(x)) \leq H(x, v(x), D^- v(x)),$$

which is exactly monotonicity of the scheme.
**Convergence rate**

\[
(S) \begin{cases}
    u_h + H(D^- u_h) = f & \text{for } x \in (0, 1]^n_h \\
    u_h = g & \text{for } x \in \Gamma_h,
\end{cases}
\]

\[
(H) \begin{cases}
    u + H(Du) = f & \text{in } (0, 1]^n \\
    u = g & \text{on } \Gamma,
\end{cases}
\]

**Theorem ([Crandall and Lions, 1984, Souganidis, 1985])**

Assume \( H \in C^{0,1}(\mathbb{R}^n) \) is directed and \( f, g \in C^{0,1}([0, 1]^n) \). Let \( u_h \) be a solution of \( S \) and let \( u \) be a viscosity solution of \( H \). If \( u \in C^{0,1}([0, 1]^n) \) and

\[
\sup_{h > 0} \sup_{x \neq y} \frac{|u^h(x) - u^h(y)|}{|x - y|} < \infty,
\]

then

\[
|u_h - u| \leq C \sqrt{h}.
\]
Sketch of proof that $u_h \leq u + C\sqrt{h}$

For $\alpha > 0$ set

$$\Phi(x, y) = u_h(x) - u(y) - \frac{\alpha}{2}|x - y|^2,$$

and let $(x_\alpha, y_\alpha) \in (0, 1]^n \times (0, 1]^n$ such that $\Phi(x_\alpha, y_\alpha) = \max[0,1]_h \times [0,1]^n \Phi$.
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Set

$$\varphi(x) = \frac{\alpha}{2}|x - y_\alpha|^2 - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 + u_h(x_\alpha).$$

Then $\varphi(x_\alpha) = u_h(x_\alpha)$ and $u_h(x) \leq \varphi(x)$ for all $x \in [0, 1]^n_h$. 
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$$\varphi(x) = \frac{\alpha}{2} |x - y_\alpha|^2 - \frac{\alpha}{2} |x_\alpha - y_\alpha|^2 + u_h(x_\alpha).$$

Then $\varphi(x_\alpha) = u_h(x_\alpha)$ and $u_h(x) \leq \varphi(x)$ for all $x \in [0, 1]^n_h$. Since $H$ is directed

$$f(x_\alpha) = u_h(x_\alpha) + H(D^- u_h(x_\alpha)) \geq u_h(x_\alpha) + H(D^- \varphi(x_\alpha)).$$
Sketch of proof that $u_h \leq u + C\sqrt{h}$

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$$\Phi(x, y) = u_h(x) - u(y) - \frac{\alpha}{2}|x - y|^2,$$

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$$f(x_\alpha) = u_h(x_\alpha) + H(D^- u_h(x_\alpha)) \geq u_h(x_\alpha) + H(D^- \varphi(x_\alpha)).$$

Since $H \in C^{0,1}(\mathbb{R}^n)$ and $D^- \varphi(x_\alpha) = p_\alpha - \frac{\alpha h}{2}(1, \ldots, 1)$ where $p_\alpha = \alpha(x_\alpha - y_\alpha)$

$$u_h(x_\alpha) + H(p_\alpha) \leq f(x_\alpha) + C\alpha h.$$
Sketch of proof that $u_h \leq u + C\sqrt{h}$

For $\alpha > 0$ set

$$
\Phi(x, y) = u_h(x) - u(y) - \frac{\alpha}{2}|x - y|^2,
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$$

Then $\varphi(x_\alpha) = u_h(x_\alpha)$ and $u_h(x) \leq \varphi(x)$ for all $x \in [0, 1]^n$. Since $H$ is directed

$$
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$$

Since $H \in C^{0,1}(\mathbb{R}^n)$ and $D^- \varphi(x_\alpha) = p_\alpha - \frac{\alpha h}{2}(1, \ldots, 1)$ where $p_\alpha = \alpha(x_\alpha - y_\alpha)$

$$
u_h(x_\alpha) + H(p_\alpha) \leq f(x_\alpha) + C\alpha h.
$$

Let $\psi(y) = \frac{\alpha}{2}|x_\alpha - y|^2$. Since $u + \psi$ has a minimum at $y_\alpha$ and $-D\psi(y_\alpha) = p_\alpha$

$$
u(y_\alpha) + H(p_\alpha) \geq f(y_\alpha).
$$
Sketch of proof that $u_h \leq u + C\sqrt{h}$

For $\alpha > 0$ set
\[
\Phi(x, y) = u_h(x) - u(y) - \frac{\alpha}{2}|x - y|^2,
\]
and let $(x_\alpha, y_\alpha) \in (0, 1)^n_h \times (0, 1)^n_h$ such that $\Phi(x_\alpha, y_\alpha) = \max_{[0,1]^n_h \times [0,1]^n_h} \Phi$.

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Then $\varphi(x_\alpha) = u_h(x_\alpha)$ and $u_h(x) \leq \varphi(x)$ for all $x \in [0, 1]^n_h$. Since $H$ is directed
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f(x_\alpha) = u_h(x_\alpha) + H(D^- u_h(x_\alpha)) \geq u_h(x_\alpha) + H(D^- \varphi(x_\alpha)).
\]

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\[
u_h(x_\alpha) + H(p_\alpha) \leq f(x_\alpha) + C\alpha h.
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Let $\psi(y) = \frac{\alpha}{2}|x_\alpha - y|^2$. Since $u + \psi$ has a minimum at $y_\alpha$ and $-D\psi(y_\alpha) = p_\alpha$
\[
u(y_\alpha) + H(p_\alpha) \geq f(y_\alpha).
\]

Hence:
\[
u_h(x_\alpha) - u(y_\alpha) \leq f(x_\alpha) - f(y_\alpha) + C\alpha h.
\]
Sketch of proof that $u_h \leq u + C\sqrt{h}$

Since $\Phi(x, y) = u_h(x) - u(y) - \frac{\alpha}{2}|x - y|^2$

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$$u_h(x_\alpha) - u(x_\alpha) \leq u_h(x_\alpha) - u(y_\alpha) - \frac{\alpha}{2} |x_\alpha - y_\alpha|^2$$

or

$$\frac{\alpha}{2} |x_\alpha - y_\alpha|^2 \leq u(x_\alpha) - u(y_\alpha) \leq C |x_\alpha - y_\alpha|.$$  

Hence $|x_\alpha - y_\alpha| \leq C/\alpha$. 
Sketch of proof that $u_h \leq u + C\sqrt{h}$

Since $\Phi(x, y) = u_h(x) - u(y) - \frac{\alpha}{2}|x - y|^2$

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Since $\Phi(x_\alpha, x_\alpha) \leq \Phi(x_\alpha, y_\alpha)$ and $u \in C^{0,1}([0, 1]^n)$

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Hence $|x_\alpha - y_\alpha| \leq C/\alpha$. Therefore

$$\max_{[0,1]^n_h}(u_h - u) \leq C\left(\frac{1}{\alpha} + \alpha h\right).$$
Sketch of proof that $u_h \leq u + C\sqrt{h}$

Since $\Phi(x, y) = u_h(x) - u(y) - \frac{\alpha}{2} |x - y|^2$

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Hence $|x_\alpha - y_\alpha| \leq C/\alpha$. Therefore

$$\max_{[0,1]^n_h} (u_h - u) \leq C \left( \frac{1}{\alpha} + \alpha h \right).$$

The best possible choice for $\alpha$ is $\alpha = 1/\sqrt{h}$ which yields

$$u_h - u \leq C\sqrt{h}. \quad \square$$
Convergence rate for (S3)

\[(P3) \prod_{i=1}^{n}(w + nx_i w_{x_i}) = f \quad \text{in } (0, 1]^n.\]

The proof that \(|w - w_h| \leq C\sqrt{h}\) is similar, with the extension property replacing the boundary condition.
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**Zeroth order term:** Notice we can write (P3) as \(H(x, u, Du) = f(x)\) where

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\[H(x, z, p) = \prod_{i=1}^{n} (z + nx_i p_i)\]

Therefore

\[
\frac{\partial}{\partial z} H(x, z, p) = H(x, z, p) \sum_{i=1}^{n} \frac{1}{z + nx_i p_i}
\]

\[
\geq n H(x, z, p) \left( \prod_{i=1}^{n} \frac{1}{z + nx_i p_i} \right)^\frac{1}{n}
\]

\[= n H(x, z, p)^{\frac{n-1}{n}}
\]

\[= n f(x)^{\frac{n-1}{n}}.
\]

Proof requires \(f \geq m > 0\).
Convergence rate for (S2)

\[
\prod_{i=1}^{n}(w + nx_i w_i) = f \quad \text{in } (0, 1]^n
\]

Recall \( v = x_1 \cdots x_n w^n \). This identity approximately holds for \( w_h \) and \( v_h \):

**Lemma**

**Assume** \( f \in C^{0,1}([0, 1]^n) \) **and** \( f > 0 \). **Then**

\[
|x_1 \cdots x_n w_h(x)^n - v_h(x)| \leq Ch \quad \text{for all } x \in [0, 1]^n_h,
\]

where

\[
C = C\left(n, [f]_{1;[0,1]^n}, \min_{[0,1]^n} f\right).
\]

The rate of convergence \( |n^n v_h(x) - u(x)^n| \leq C \sqrt{h} \) follows from the lemma and the rate \( \nabla w_h(x) - w(x) | \leq C \sqrt{h} \).
Outline

1 Introduction
   - Motivating example: Image retrieval
   - Non-dominated sorting

2 Continuum limit for nondominated sorting
   - Hamilton-Jacobi equation for layers
   - PDE-based ranking

3 Numerical schemes
   - An $O(h^{1/n})$ scheme
   - Two (formally) $O(h)$ schemes
   - Regularity
   - Convergence rates

4 Experimental results

5 References
We consider 3 test cases:

\[ u_1(x) = n \max_{i \in \{1, \ldots, n\}} \left\{ \left( x_i - \frac{1}{2} \right) + \prod_{j \neq i} x_j \right\} \]

\[ u_2(x) = \frac{1}{k+1} (x_1 \cdots x_n)^{\frac{1}{n}} \left( \sum_{j=1}^{n} \sin(kx_j)^2 + nk \right) \quad (k = 20) \]

\[ u_3(x) = n (x_1 \cdots x_n)^{\frac{1}{n}} \left( C \max\{x_1, \ldots, x_n\} + \sum_{j=1}^{n} x_j \right) \quad (C = 10) \]
<table>
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<tr>
<th>Mesh size $h$</th>
<th>(S1) $\ell^\infty$ Error</th>
<th>Order</th>
<th>(S2) $\ell^\infty$ Error</th>
<th>Order</th>
<th>(S3) $\ell^\infty$ Error</th>
<th>Order</th>
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Outline

1 Introduction
   - Motivating example: Image retrieval
   - Non-dominated sorting

2 Continuum limit for nondominated sorting
   - Hamilton-Jacobi equation for layers
   - PDE-based ranking

3 Numerical schemes
   - An $O(h^{1/n})$ scheme
   - Two (formally) $O(h)$ schemes
   - Regularity
   - Convergence rates

4 Experimental results

5 References


A Hamilton-Jacobi equation for the continuum limit of non-dominated sorting.

A PDE-based approach to nondominated sorting.

Two approximations of solutions of Hamilton-Jacobi equations.

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