Numerical Schemes for the Hamilton-Jacobi Equation
Continuum Limit of Non-dominated Sorting

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Applied Mathematics Seminar

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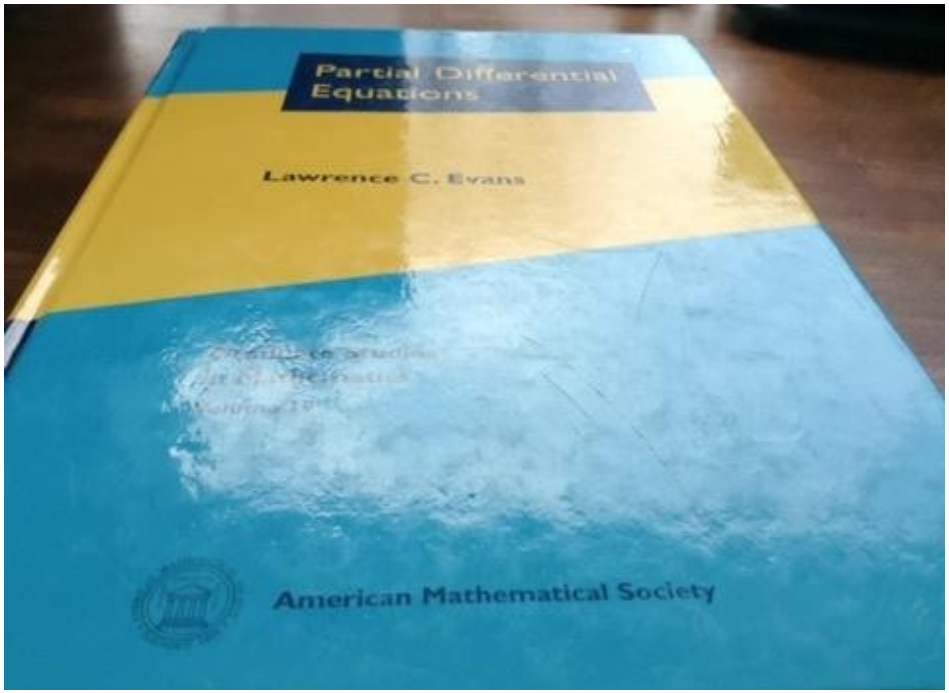
Outline

Motivating example: Google Goggles



Query image

Retrieved images

Multi-query image retrieval

Problem: Find images in a dataset S that are similar to multiple query images.

Pareto method: Solve the multi-criteria optimization problem

$$\arg\min_{I \in S}(\text{dist}(I, Q_1), \ldots, \text{dist}(I, Q_d)).$$

Query 1

Query 2

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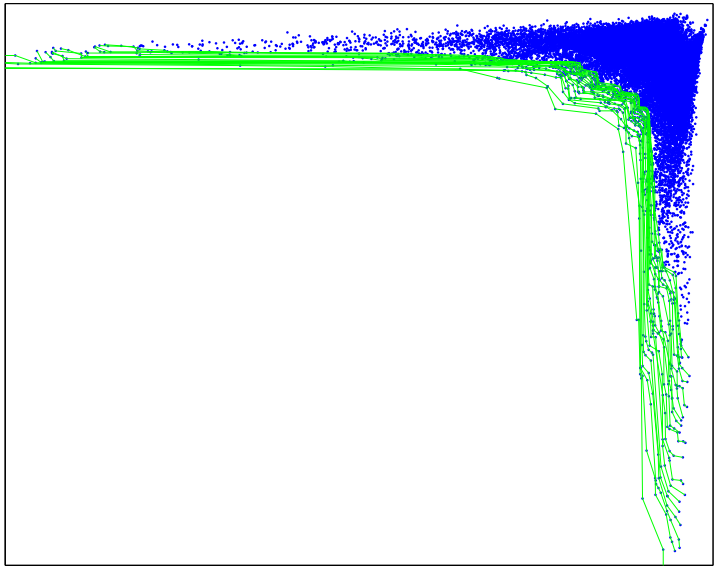
Query 2

Pareto points:



Multi-objective optimization

How do we solve the multi-objective optimization problem

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Basic approach:

1. Choose some weights $\alpha_i \in [0, 1]$ with $\sum_{i=1}^{d} \alpha_i = 1$ and define

$$f_\alpha(I) = \alpha_1 f_1(I) + \alpha_2 f_2(I) + \cdots + \alpha_d f_d(I).$$

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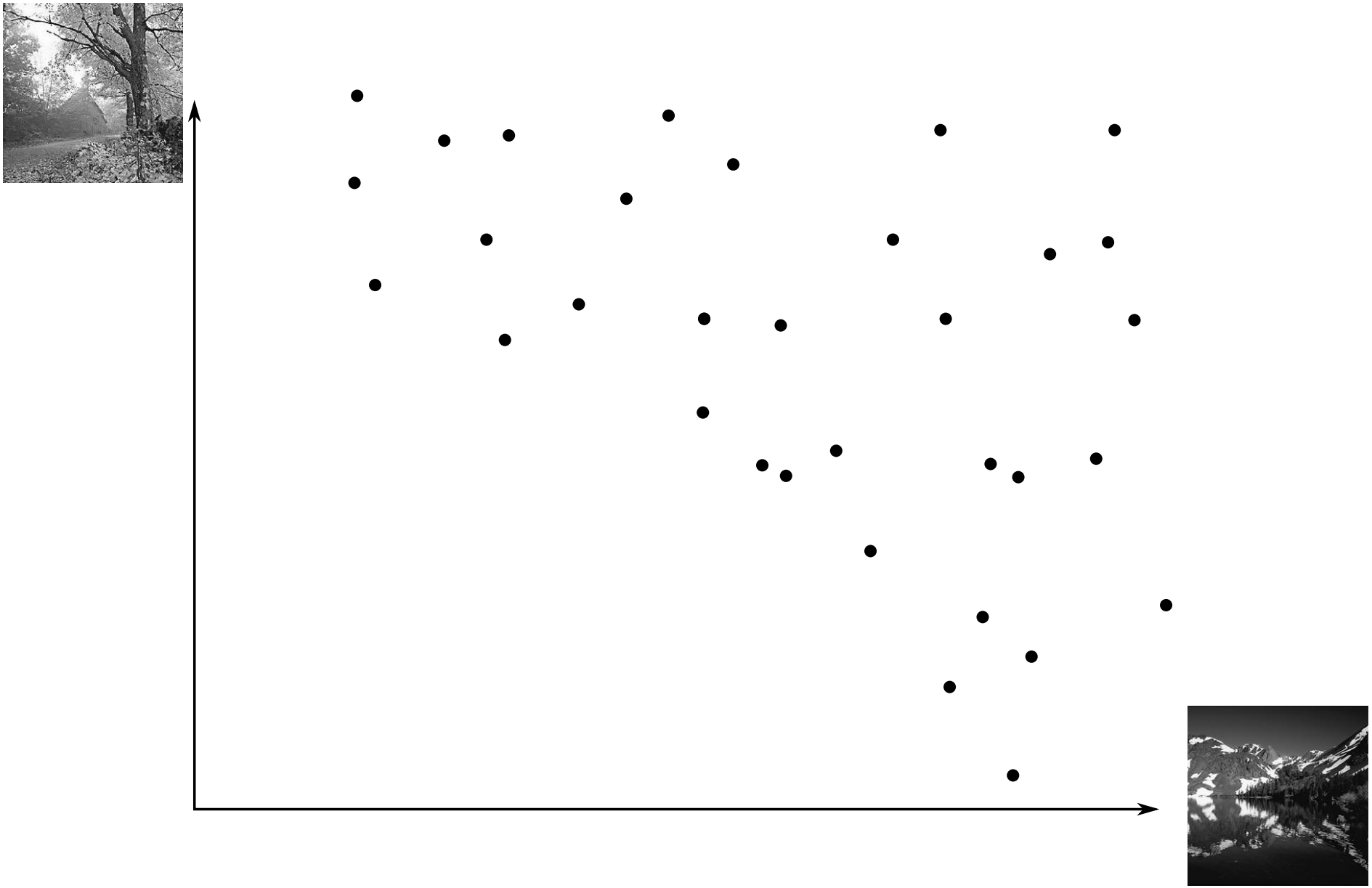
Problems:

1. Difficult to choose weights
2. Ignores relevant solutions

Basic approach



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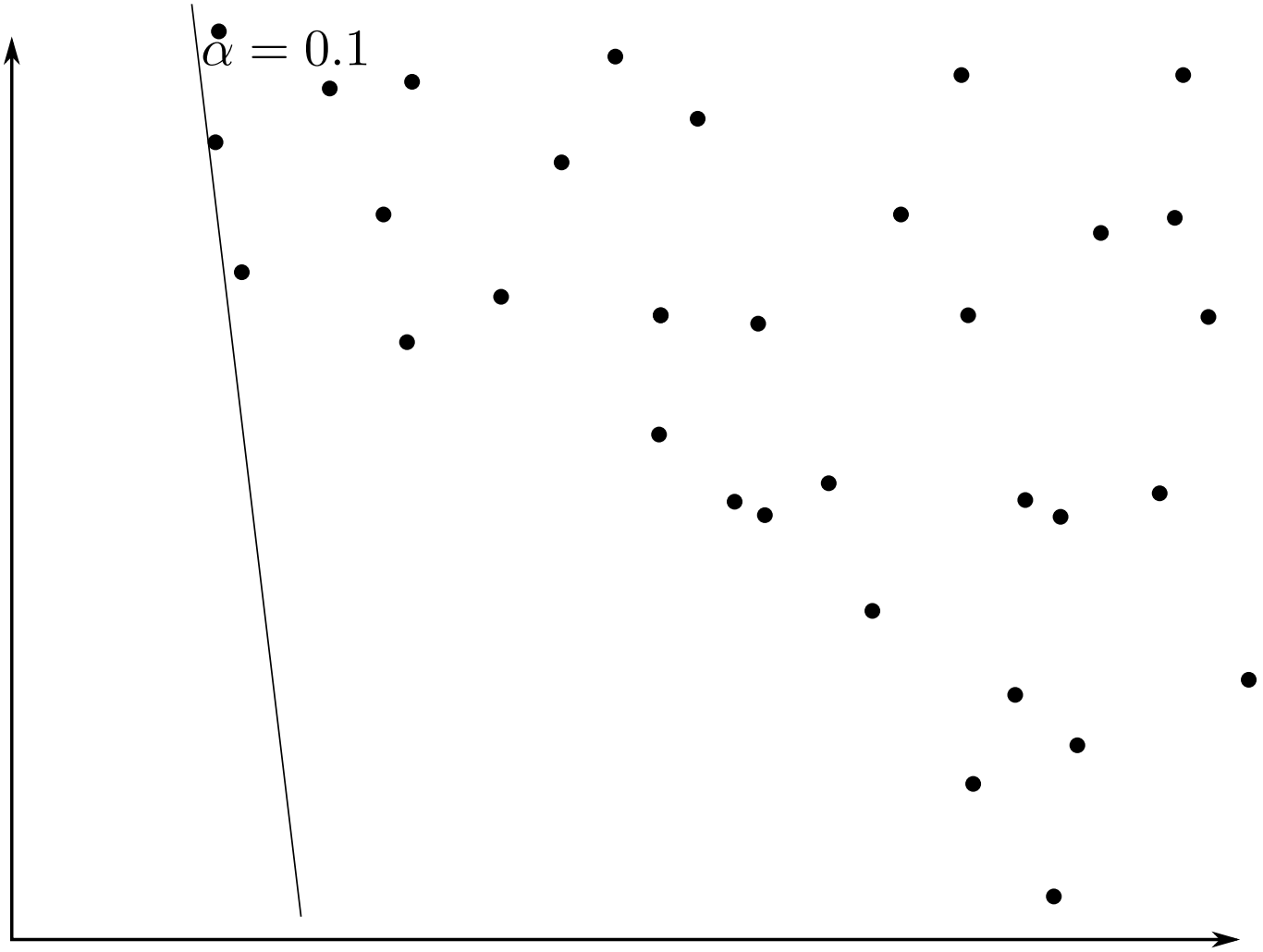

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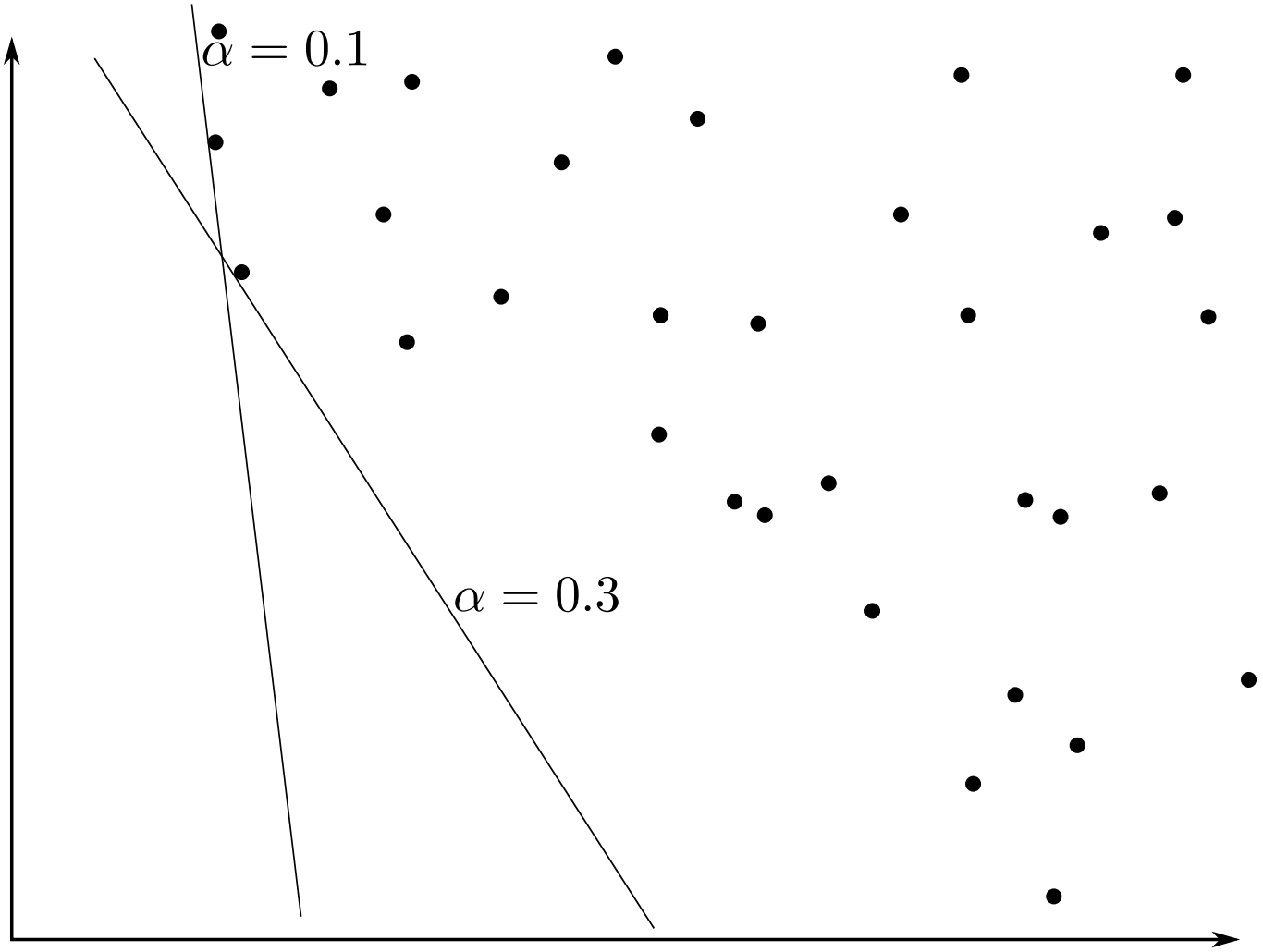

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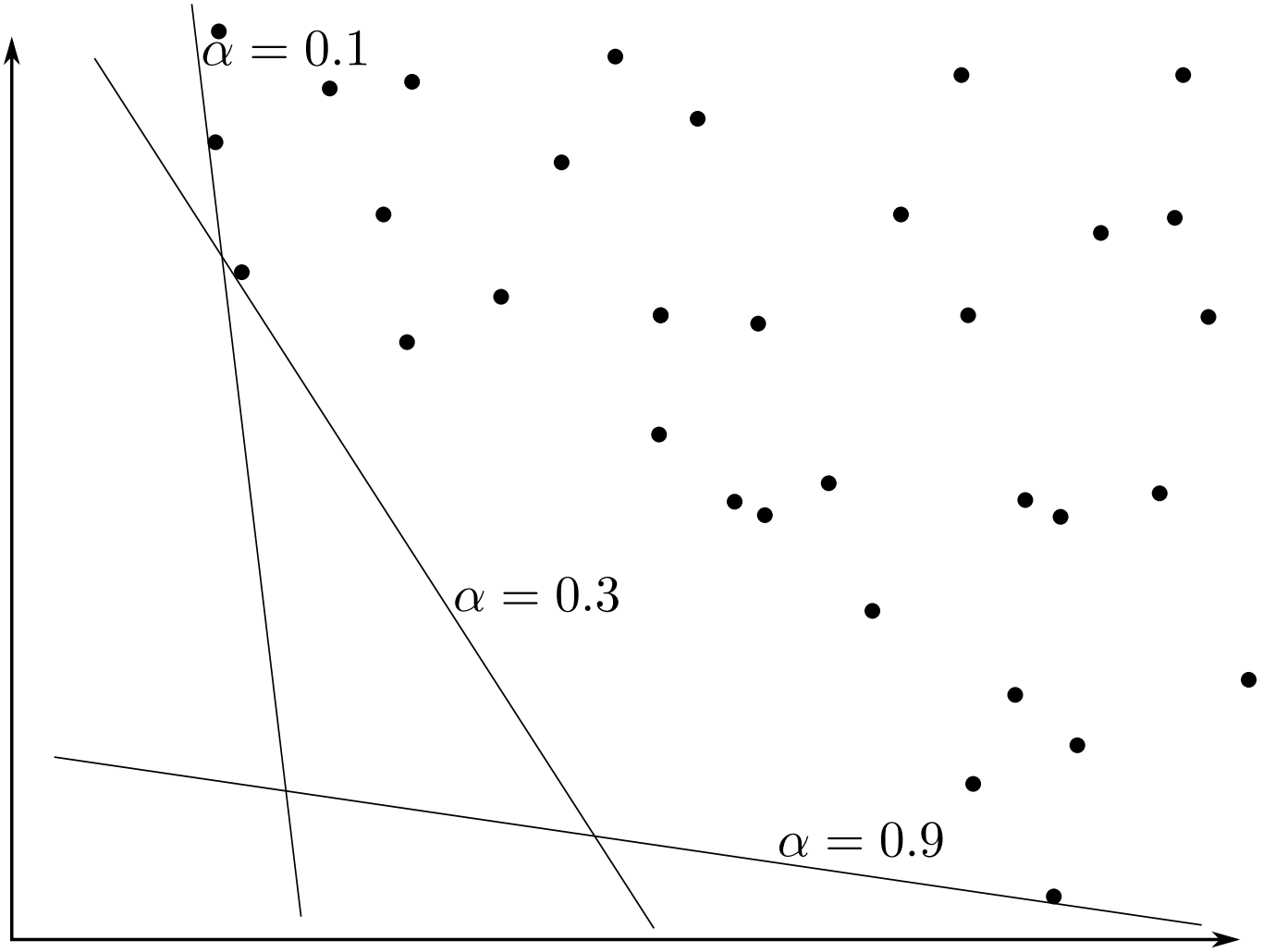
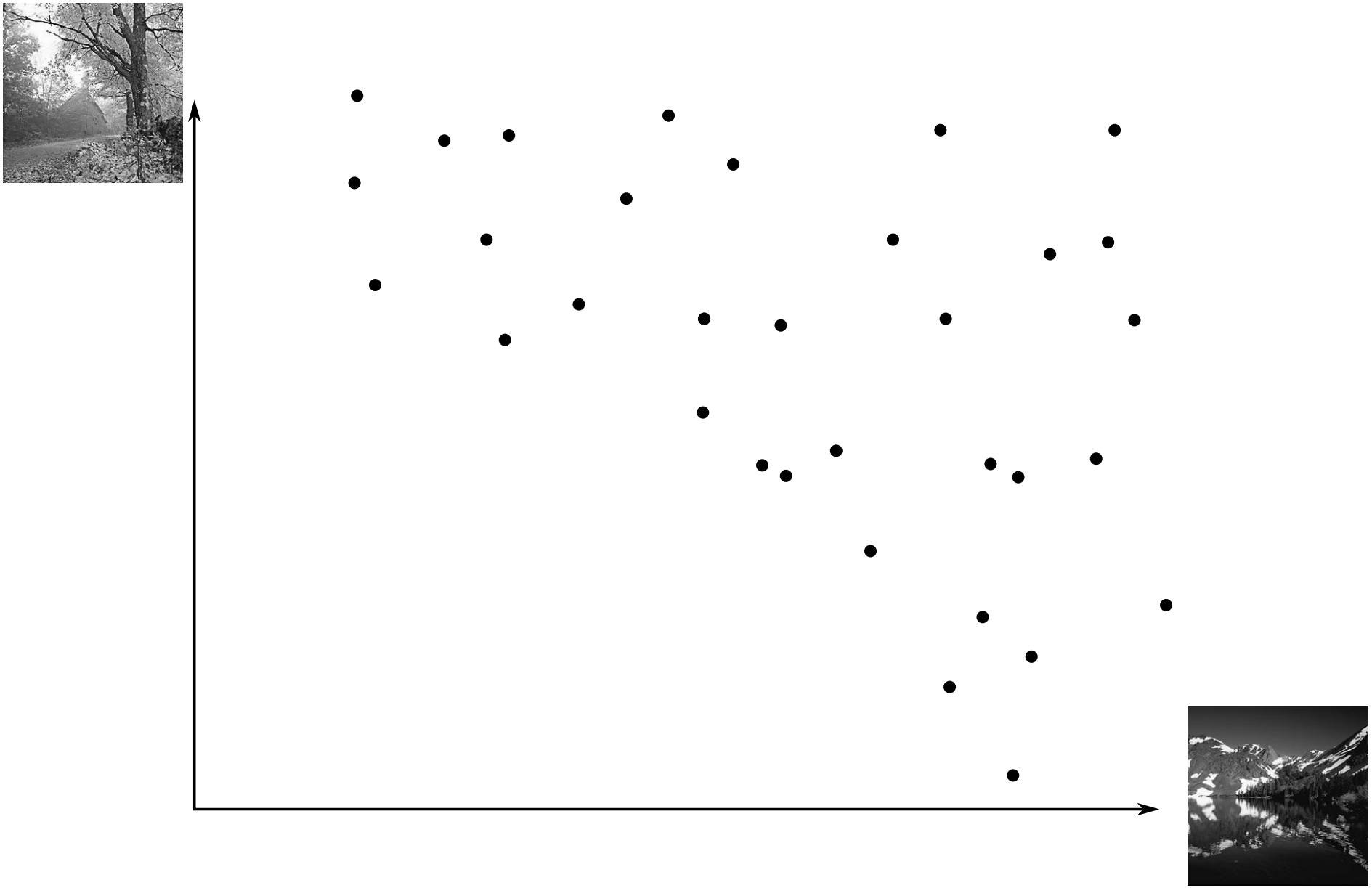


Non-dominated solutions



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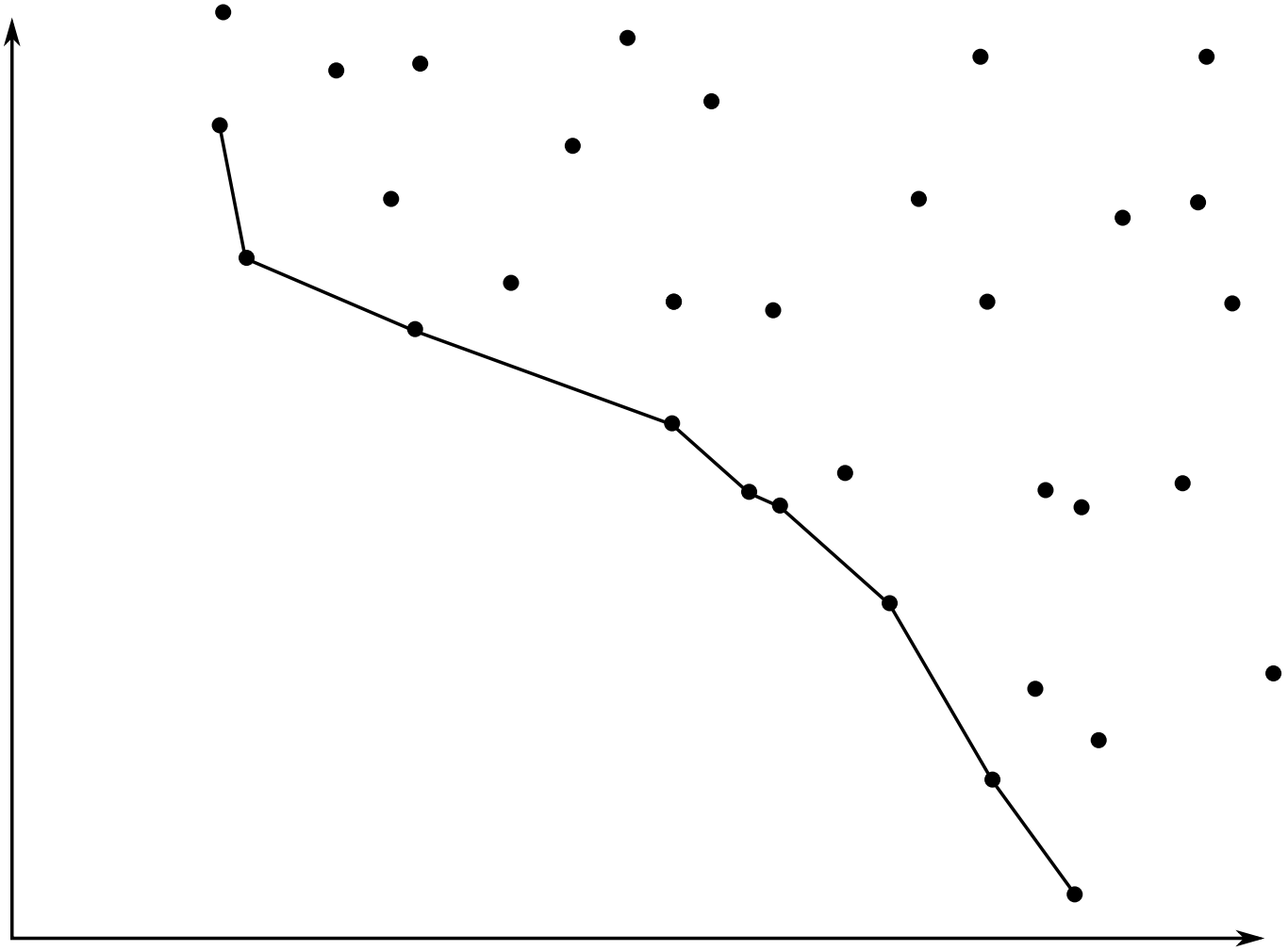

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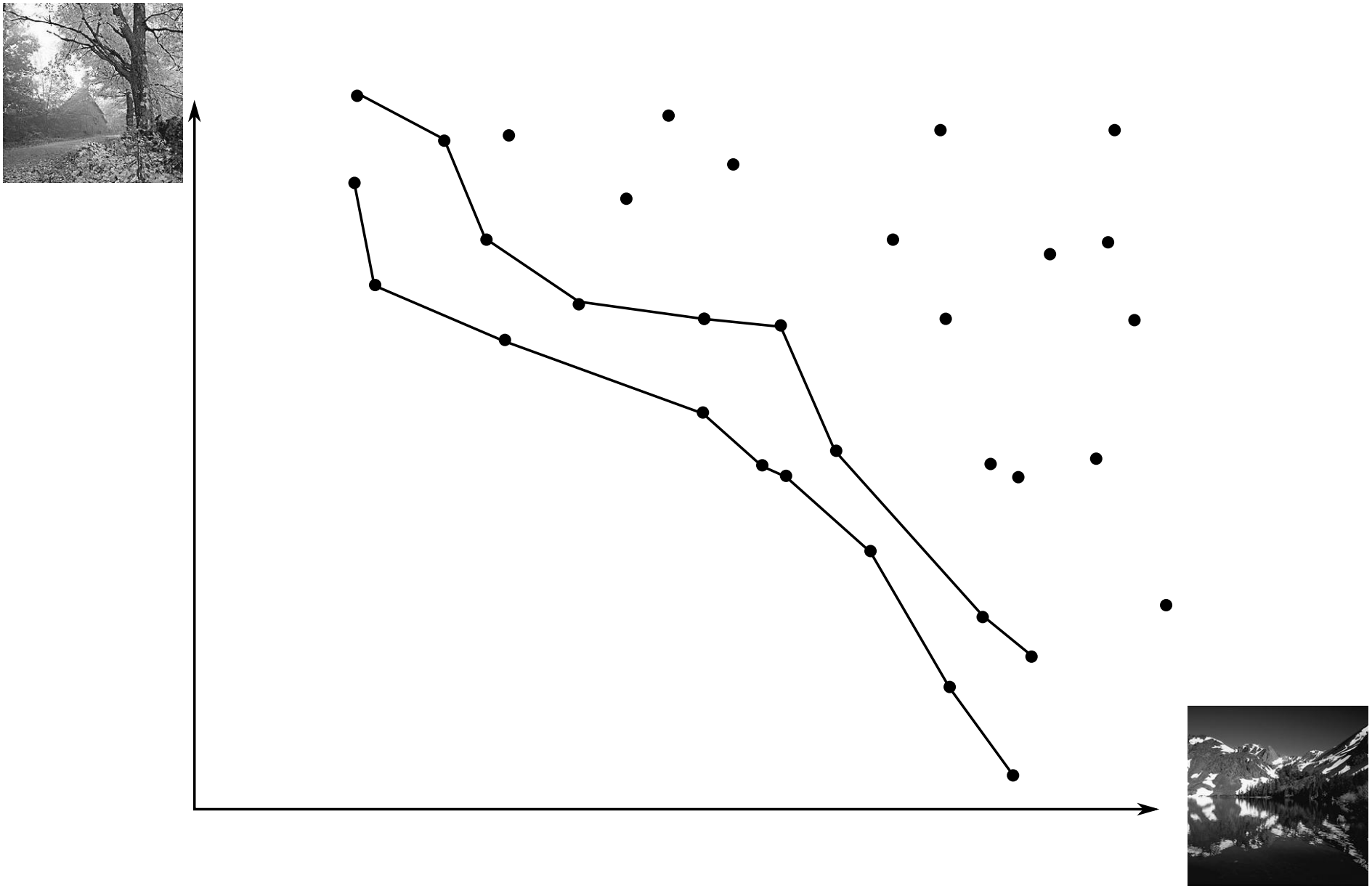

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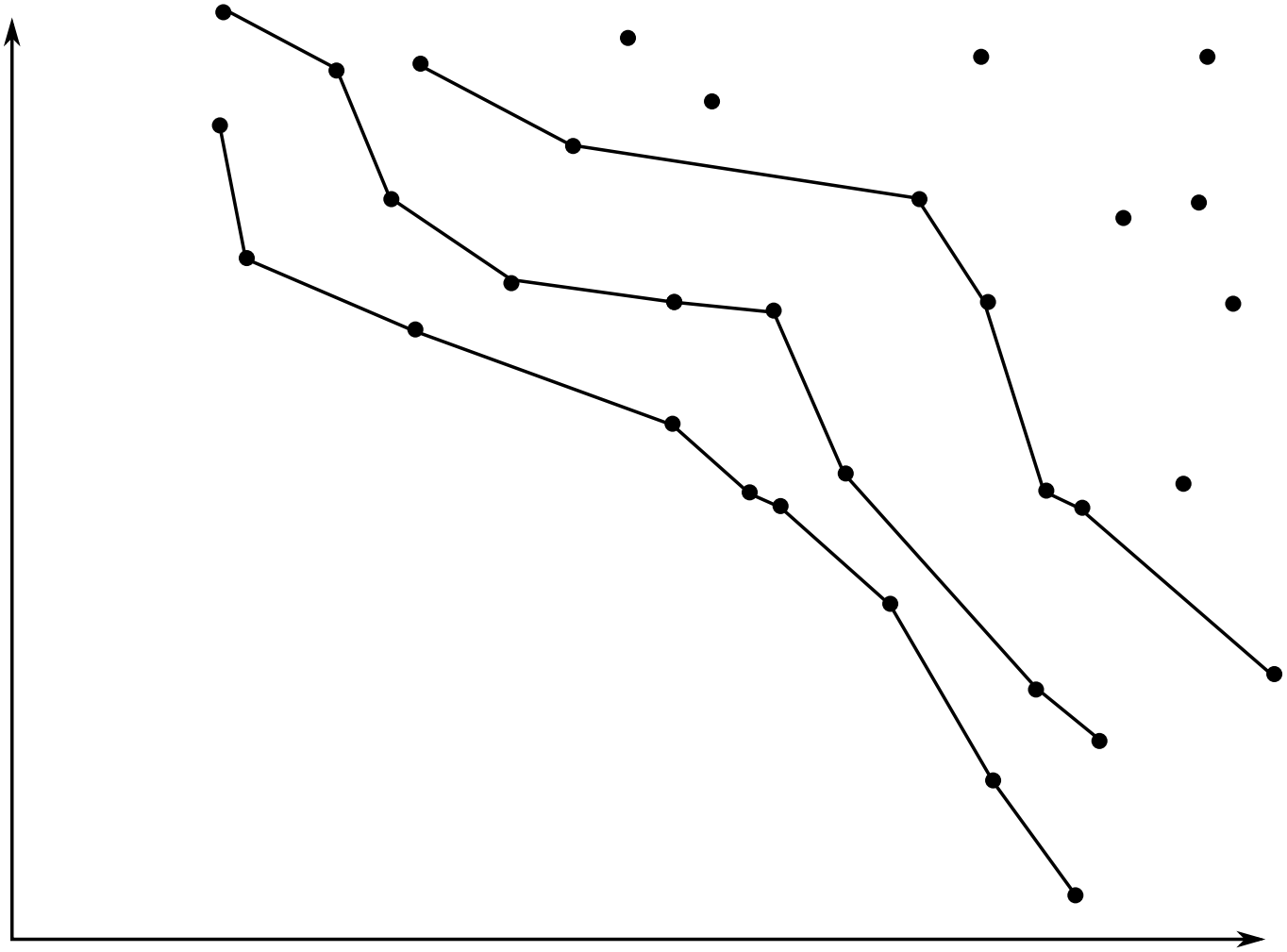

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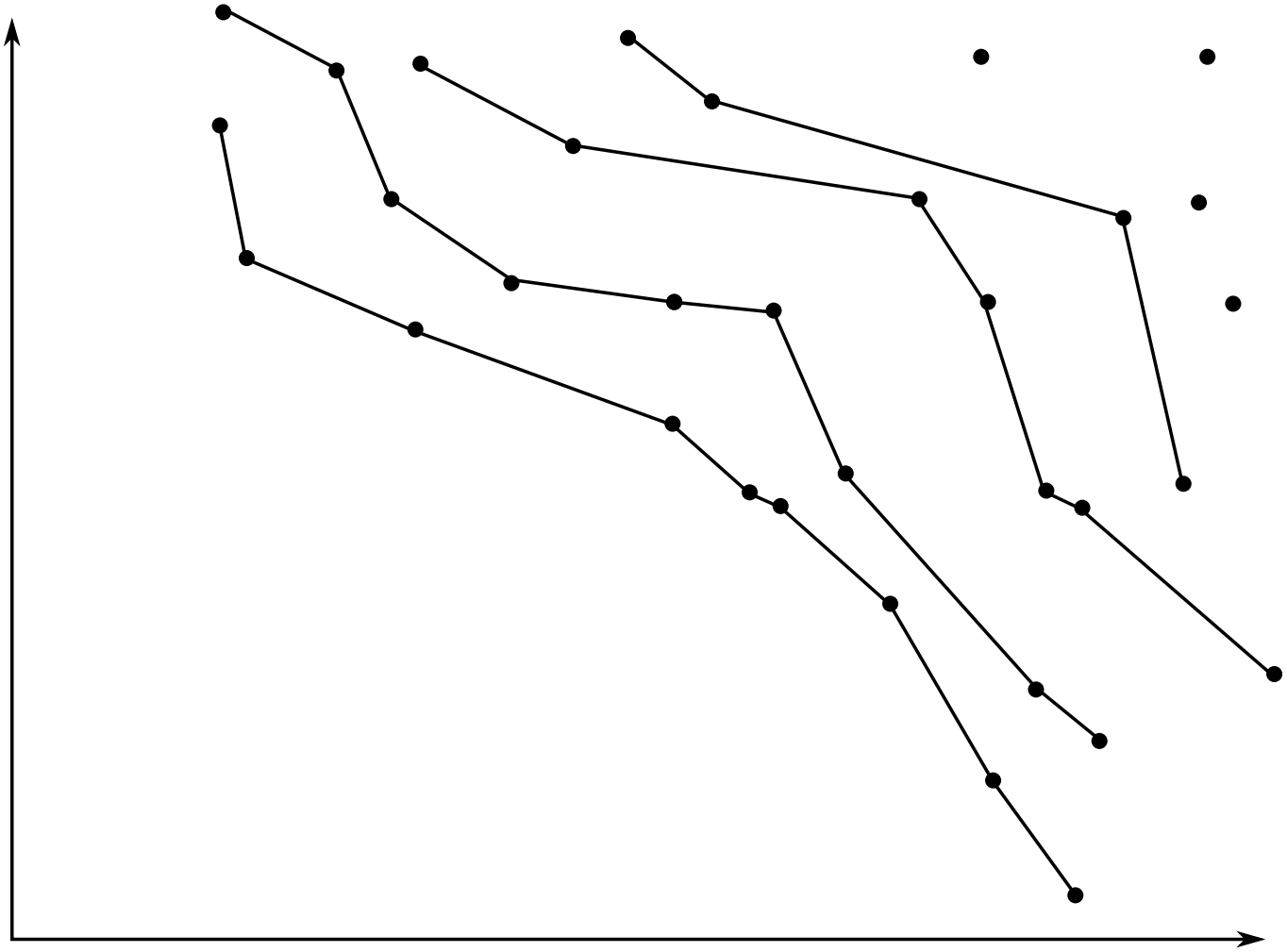

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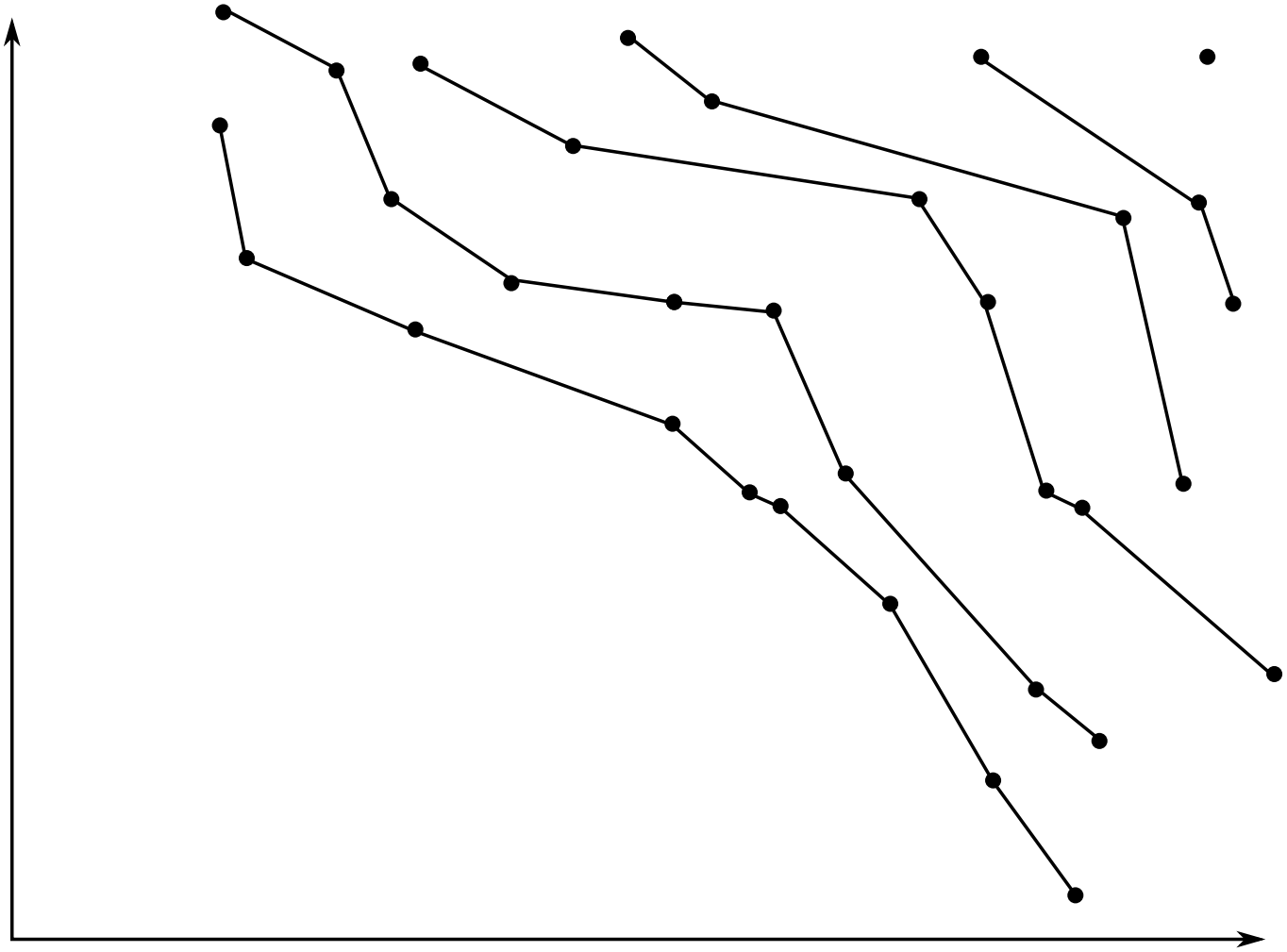

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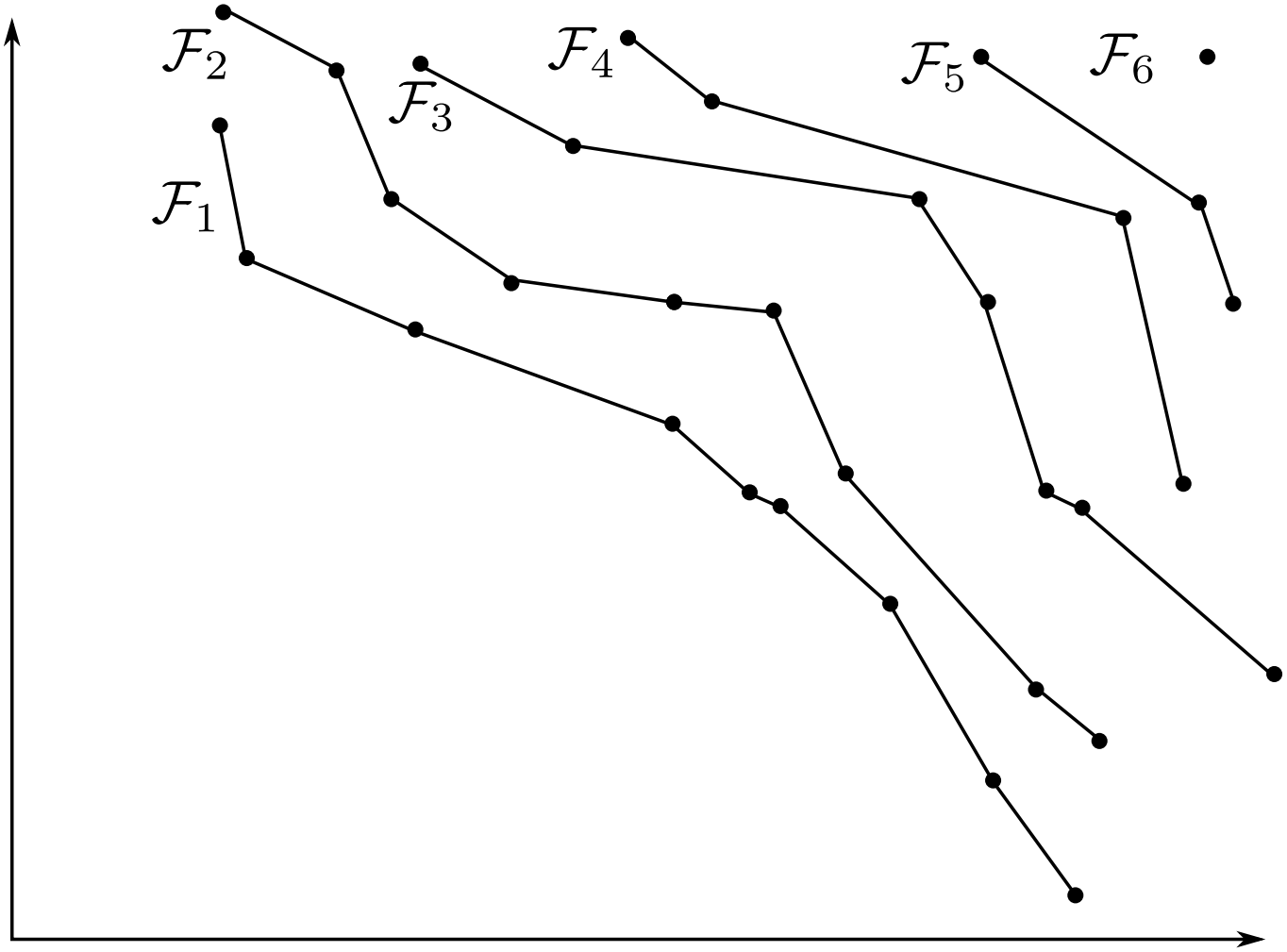
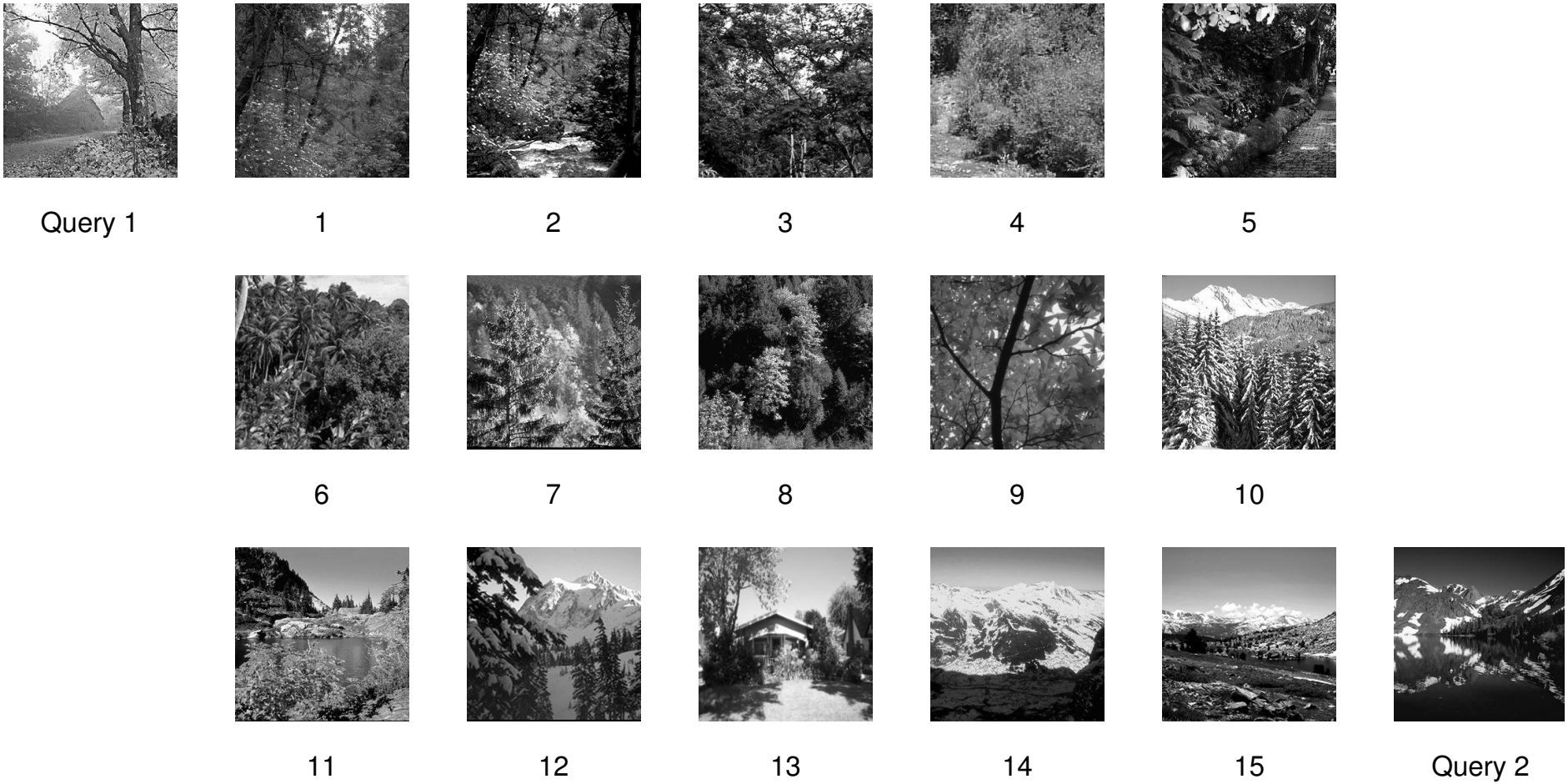


Multi-query image retrieval

First Pareto front:



Hsiao, K.-J., Calder, J., and Hero III, A. O. (2015). Pareto-depth for multiple-query image retrieval. *IEEE Transactions on Image Processing*, 24(2):583–594.

Non-dominated sorting

Let $X_1, \ldots, X_N$ be points in $\mathbb{R}^n$ and set $S = \{X_1, \ldots, X_N\}$.

Define the partial order

$$x \leqq y \iff x_i \leq y_i \text{ for all } i \in \{1, \ldots, n\}.$$

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Definition

Non-dominated sorting is the process of arranging S into layers $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots,$ defined by

$$\mathcal{F}_1 = \text{Minimal elements of } S,$$

$$\mathcal{F}_k = \text{Minimal elements of } S \setminus \bigcup_{j \leq k-1} \mathcal{F}_j.$$

Applications

Multi-objective optimization

- Genetic algorithms [Deb et al., 2002]
- Gene selection and ranking [Hero, 2003]
- Database systems [Papadias et al., 2005]
- Anomaly detection [Hsiao et al., 2012, Hsiao et al., 2015b]
- Image retrieval [Hsiao et al., 2015a]

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Combinatorics and probability

- Longest chain in Euclidean space [Hammersley, 1972]
- Patience sorting [Aldous and Diaconis, 1999]
- Young Tableaux [Viennot, 1984]
- Graph theory [Lou and Sarrafzadeh, 1993]
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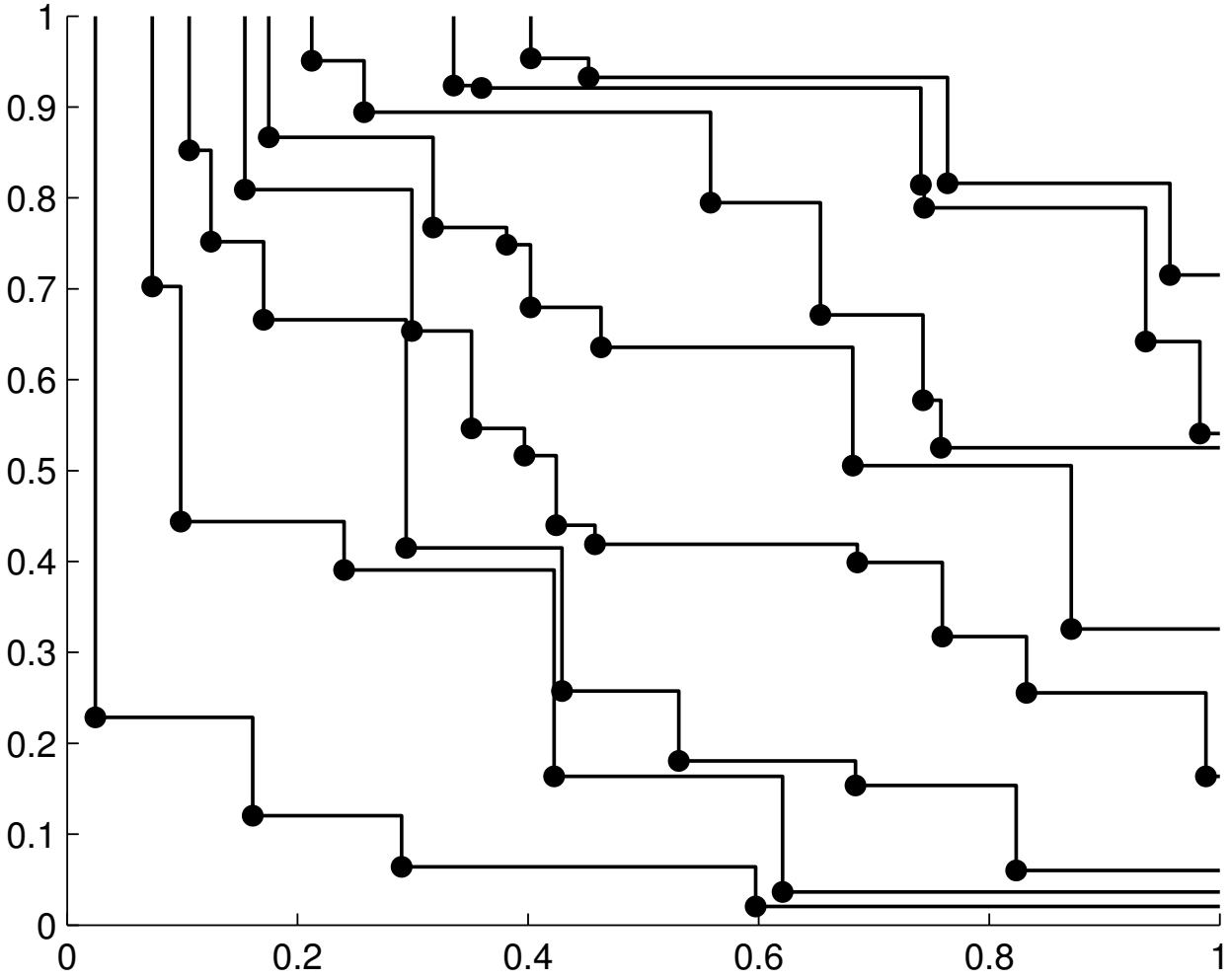
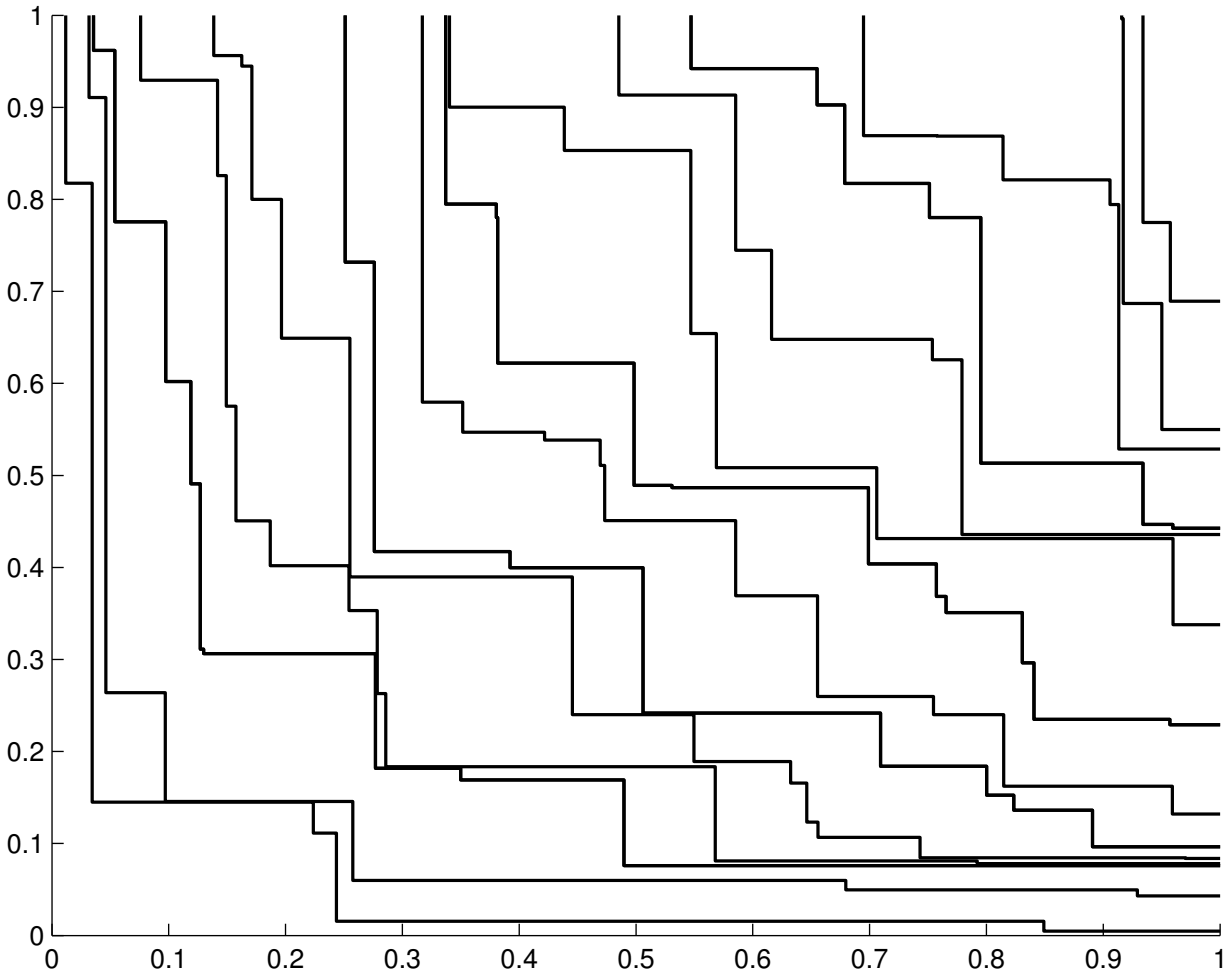
Other applications

- Molecular biology [Pevzner, 2000]
- Integrated circuit design [Adhar, 2007]

Demo: 50 Random samples

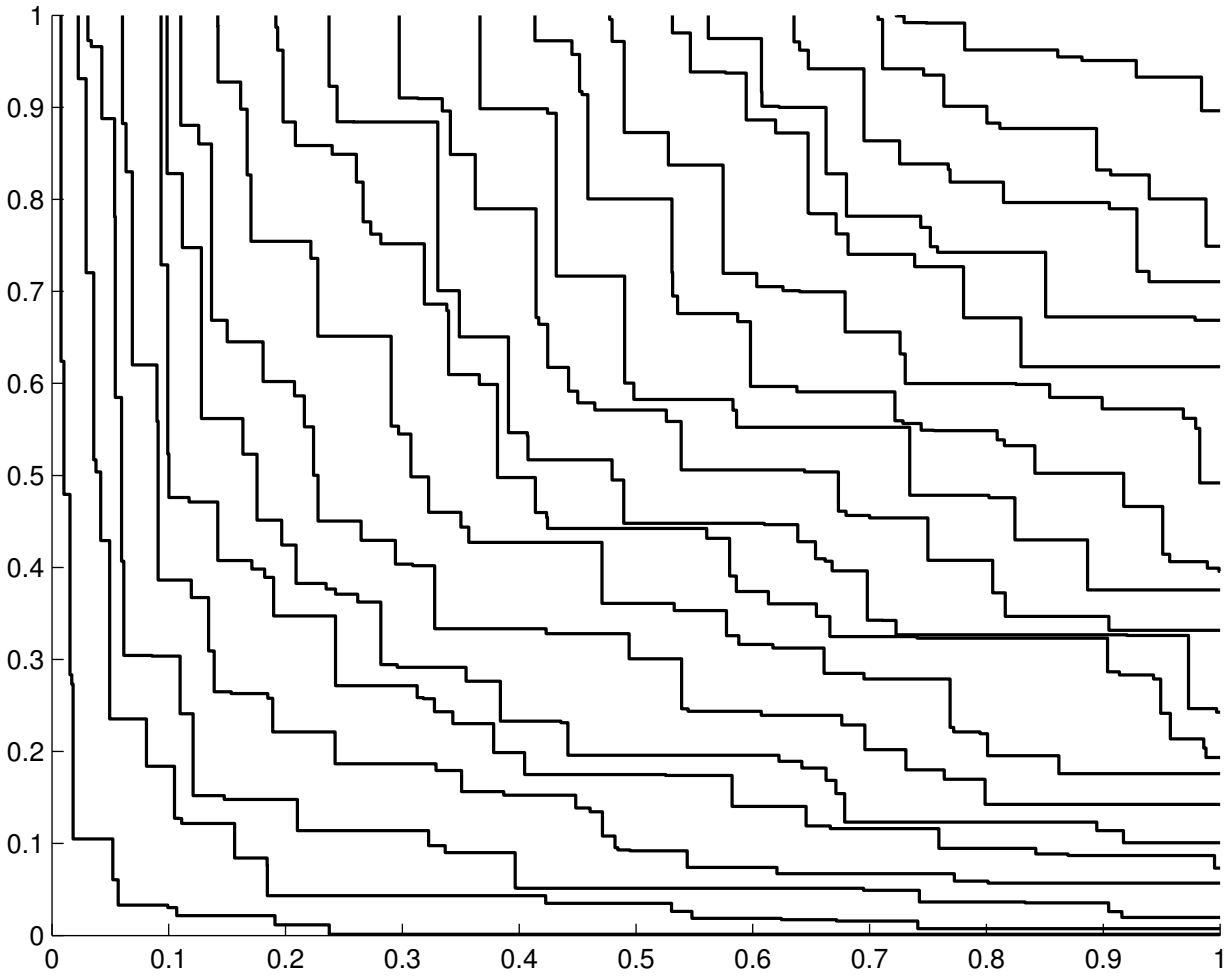


Demo: Uniform distribution



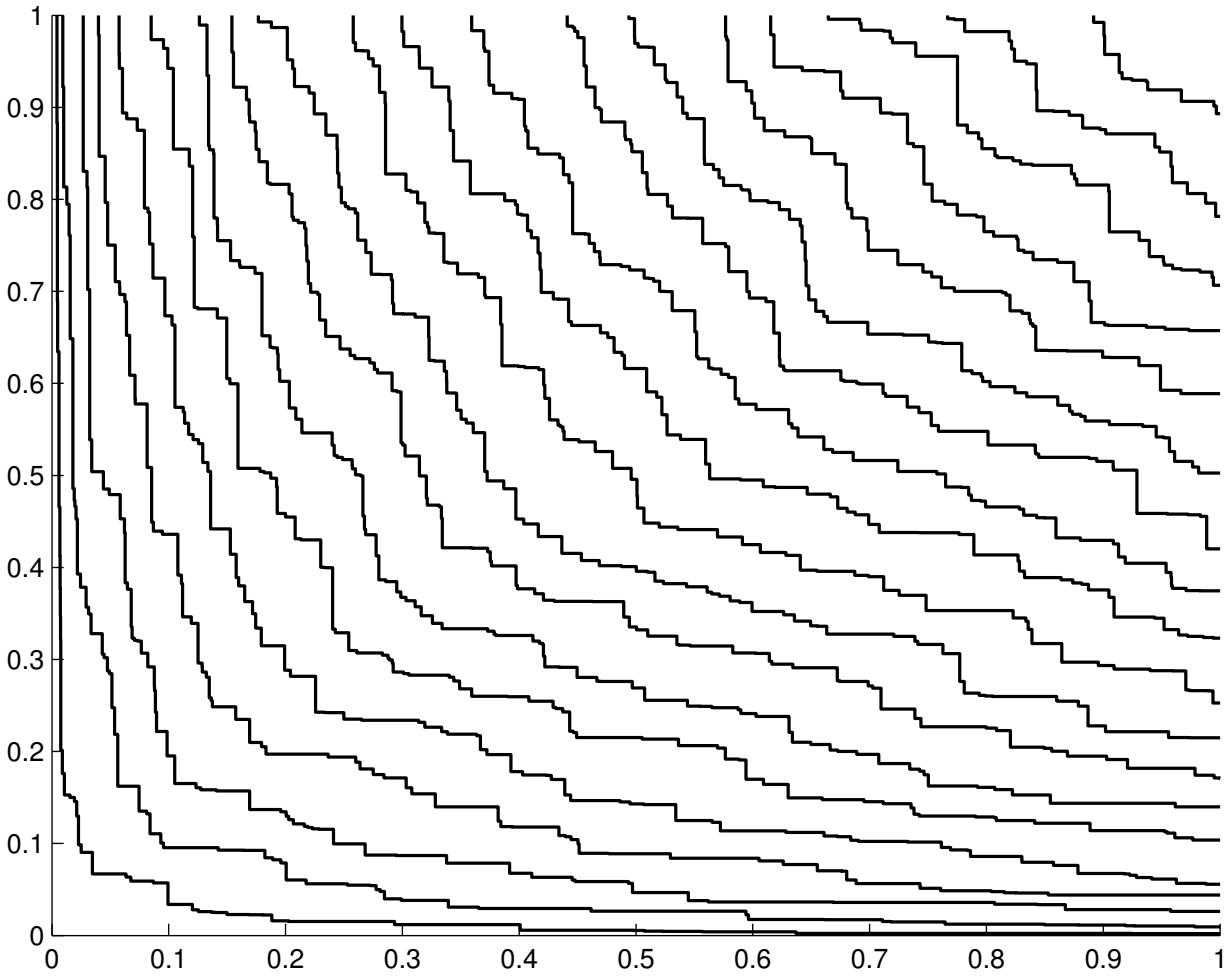
$N = 10^2$ points

Demo: Uniform distribution



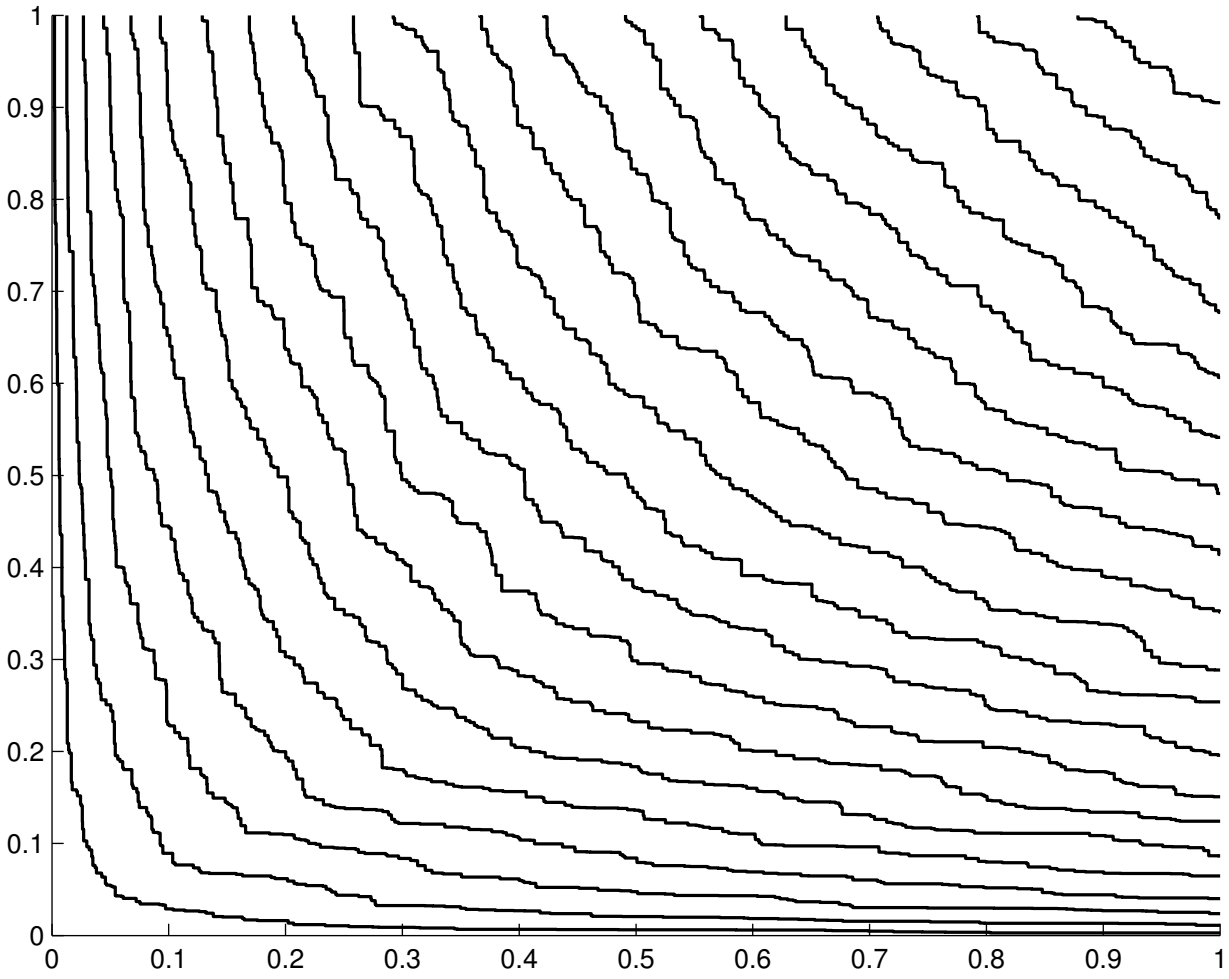
$N = 10^3$ points

Demo: Uniform distribution



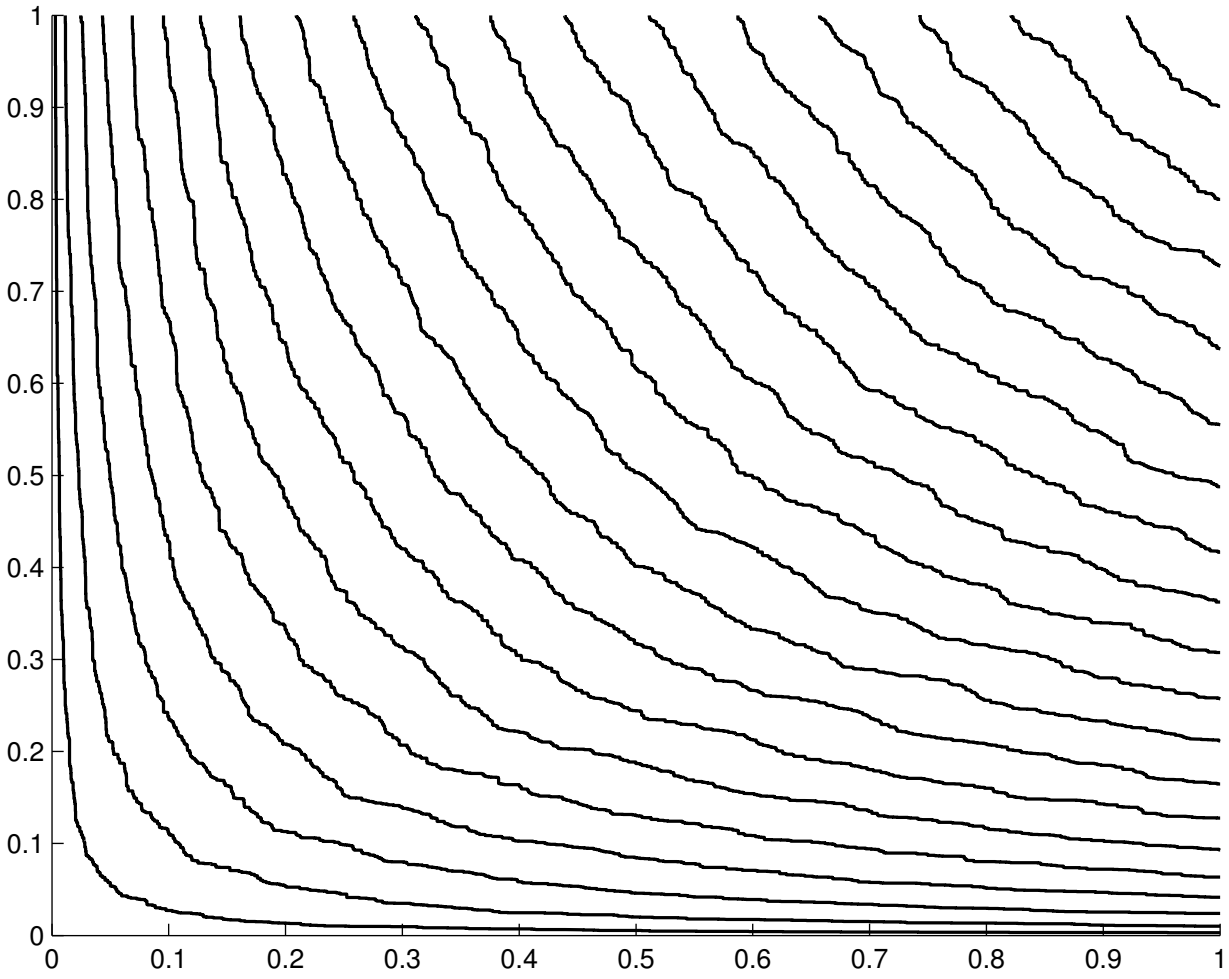
$N = 10^4$ points

Demo: Uniform distribution



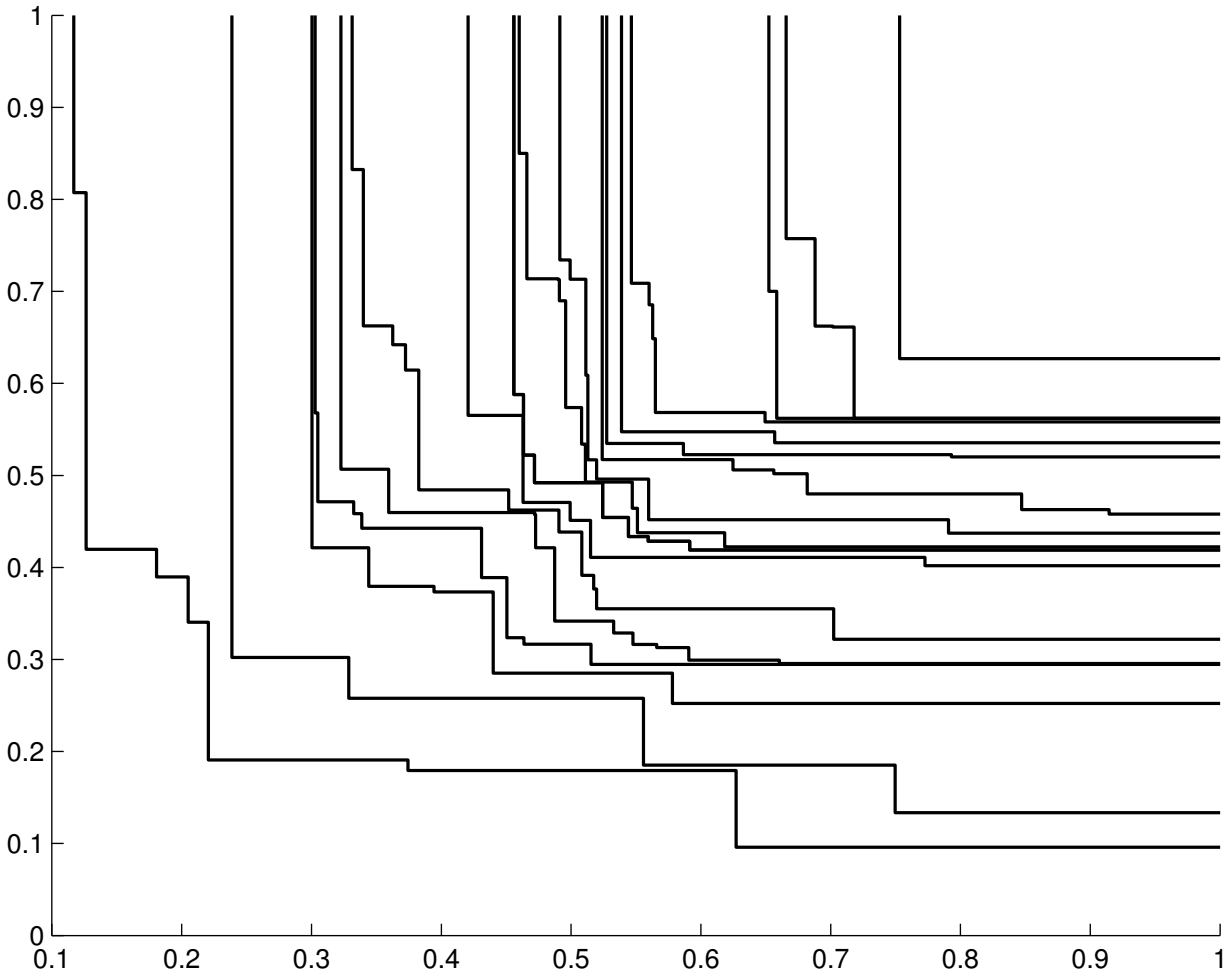
$N = 10^5$ points

Demo: Uniform distribution



$N = 10^6$ points

Demo: Gaussian distribution



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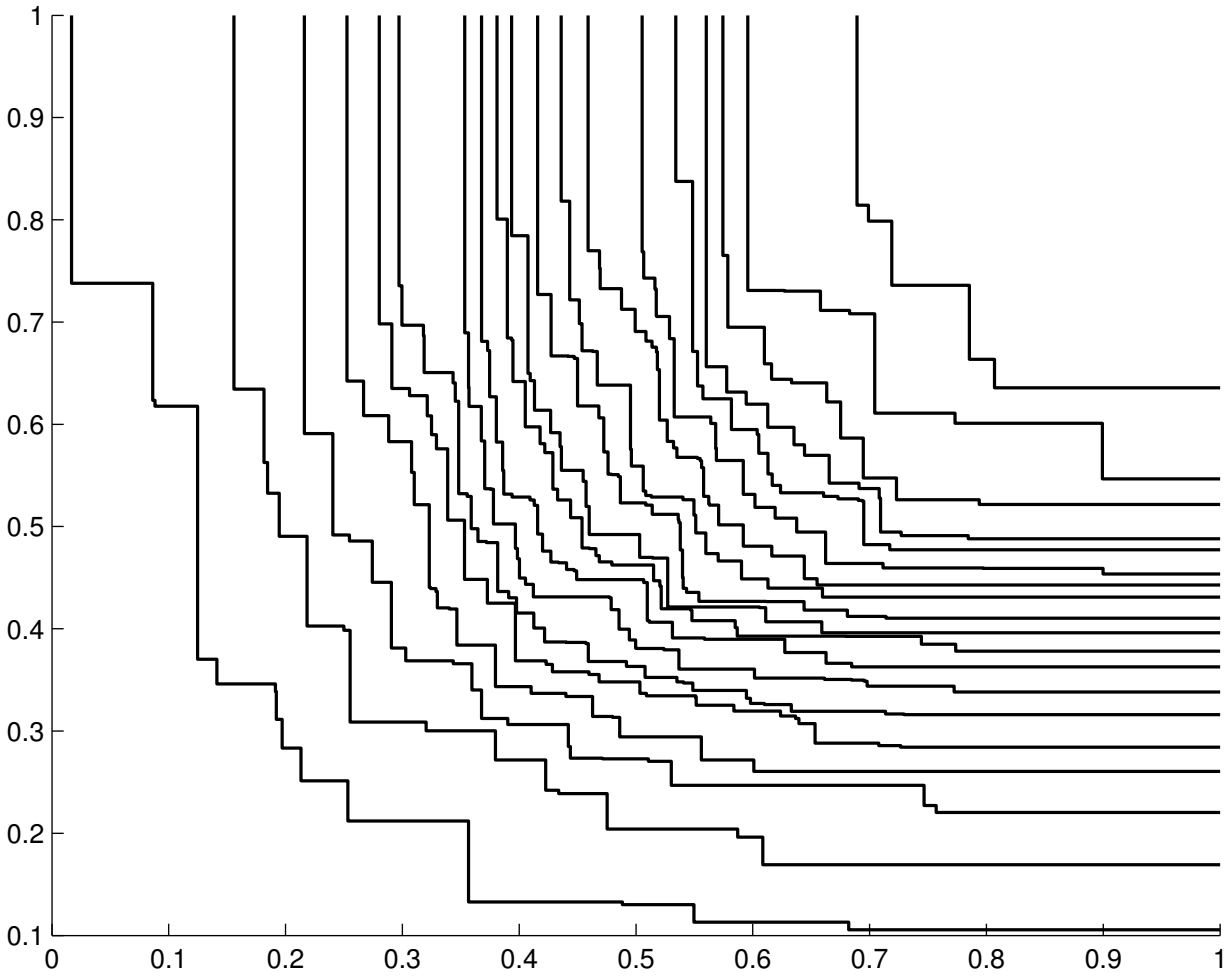

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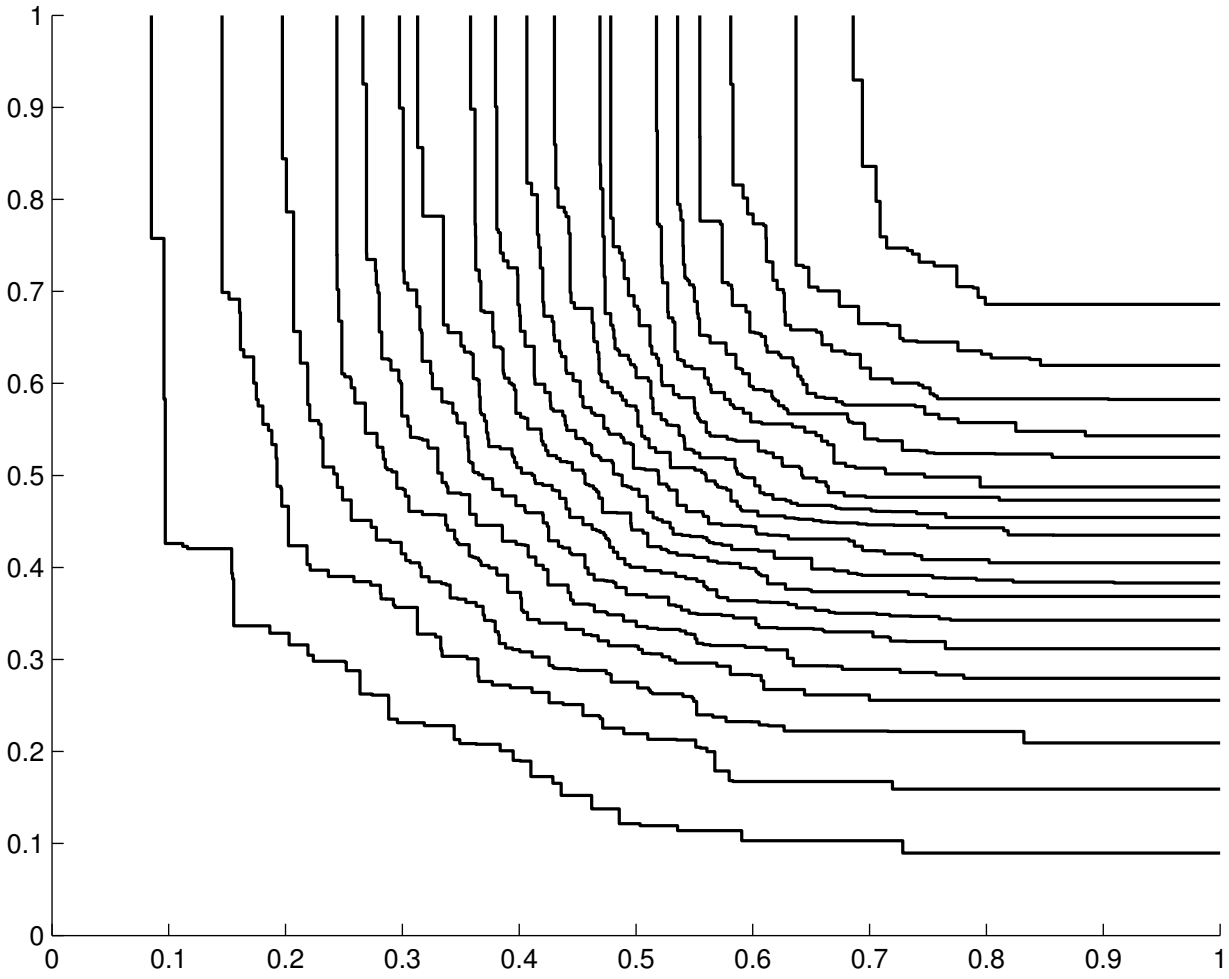

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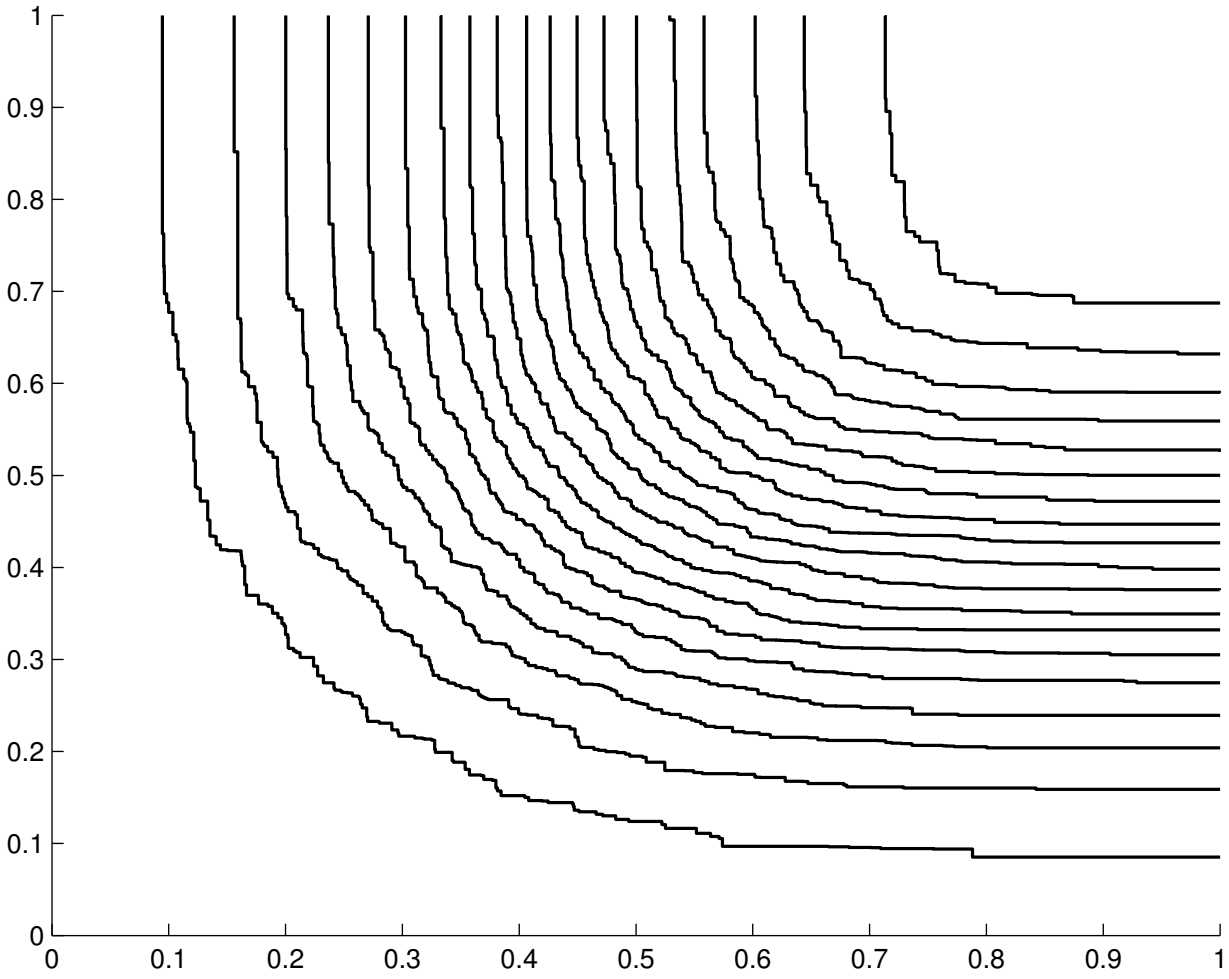

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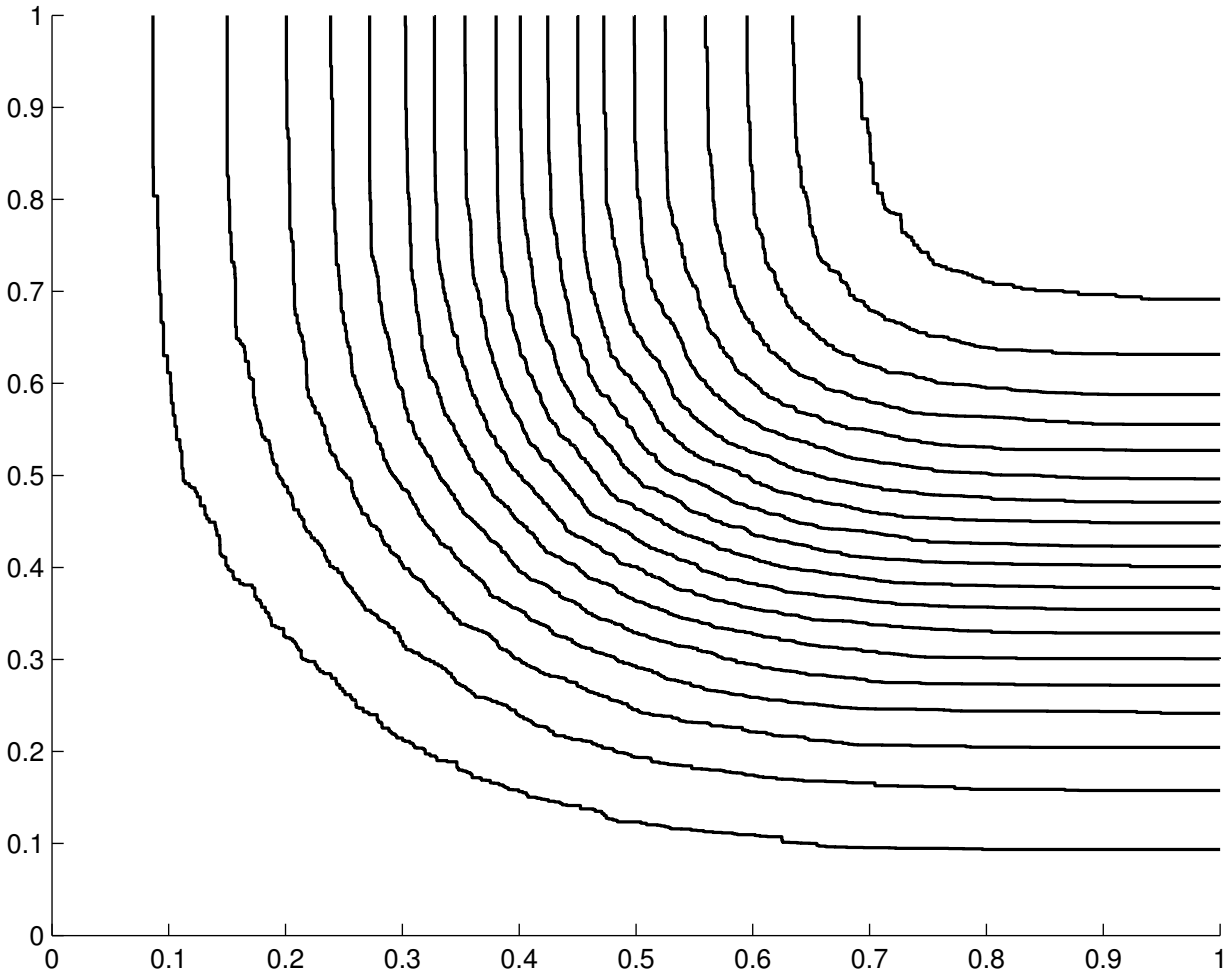
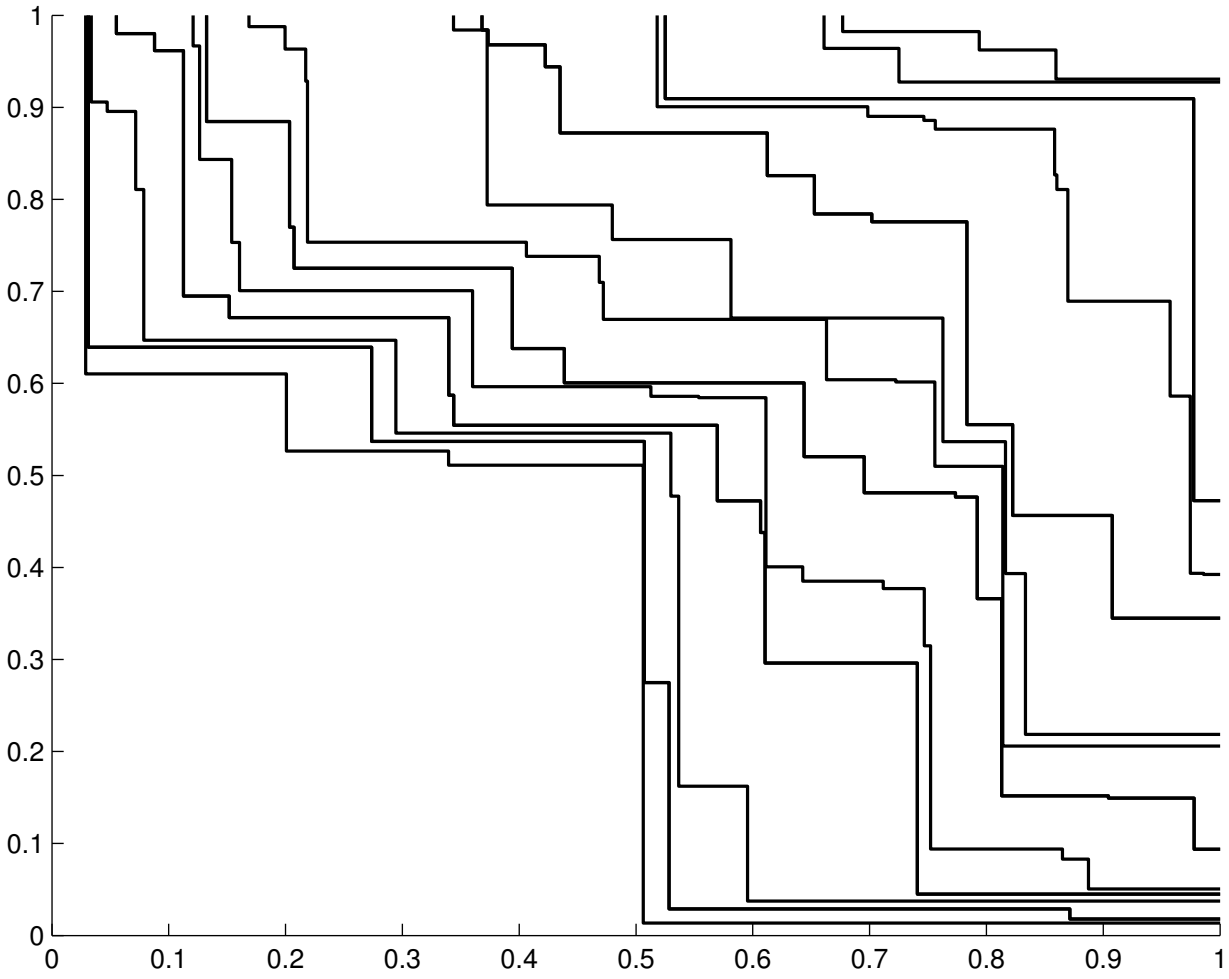


$N = 10^6$ points

Demo: Uniform distribution on $[0,1]^2 \setminus [0,0.5]^2$



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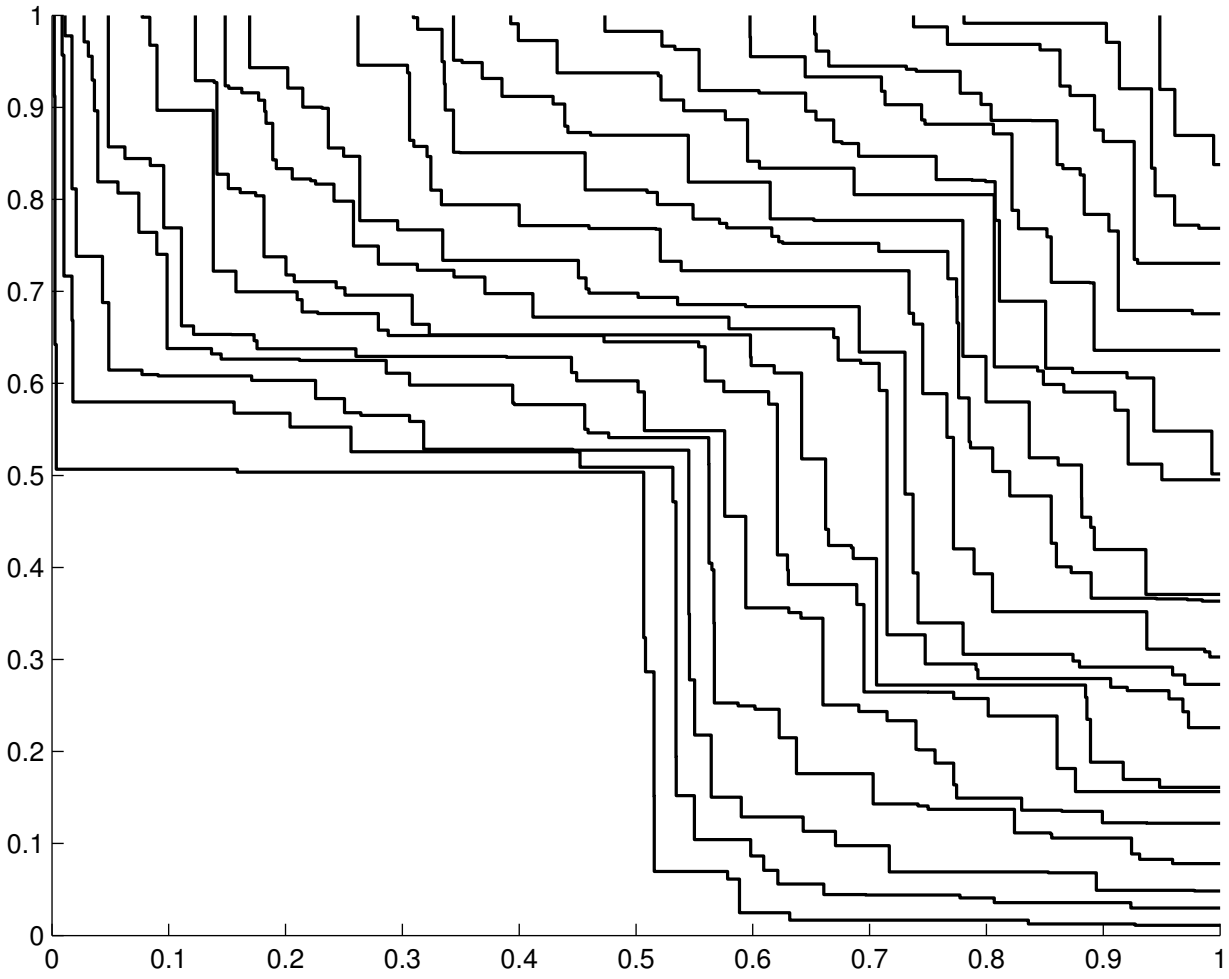

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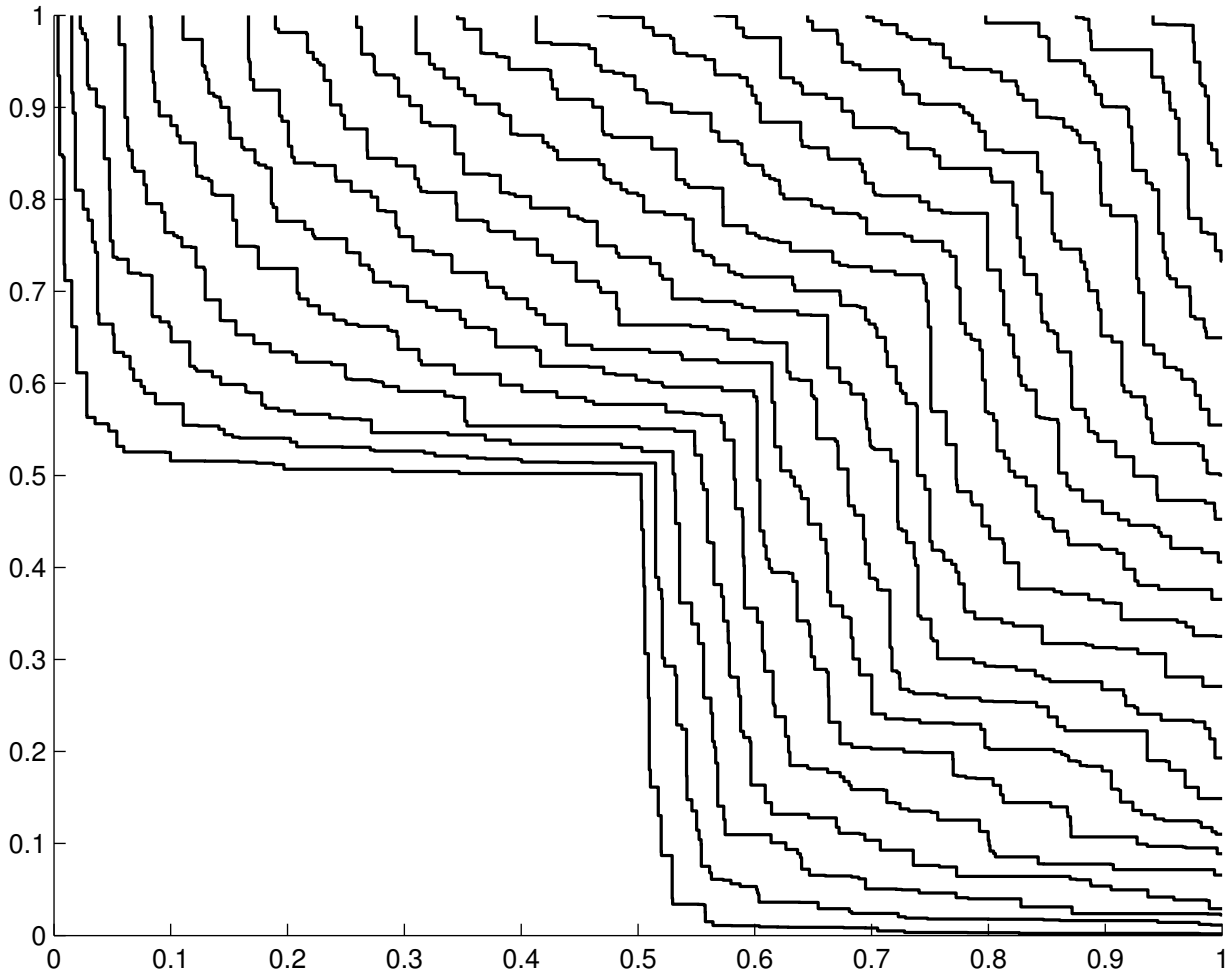

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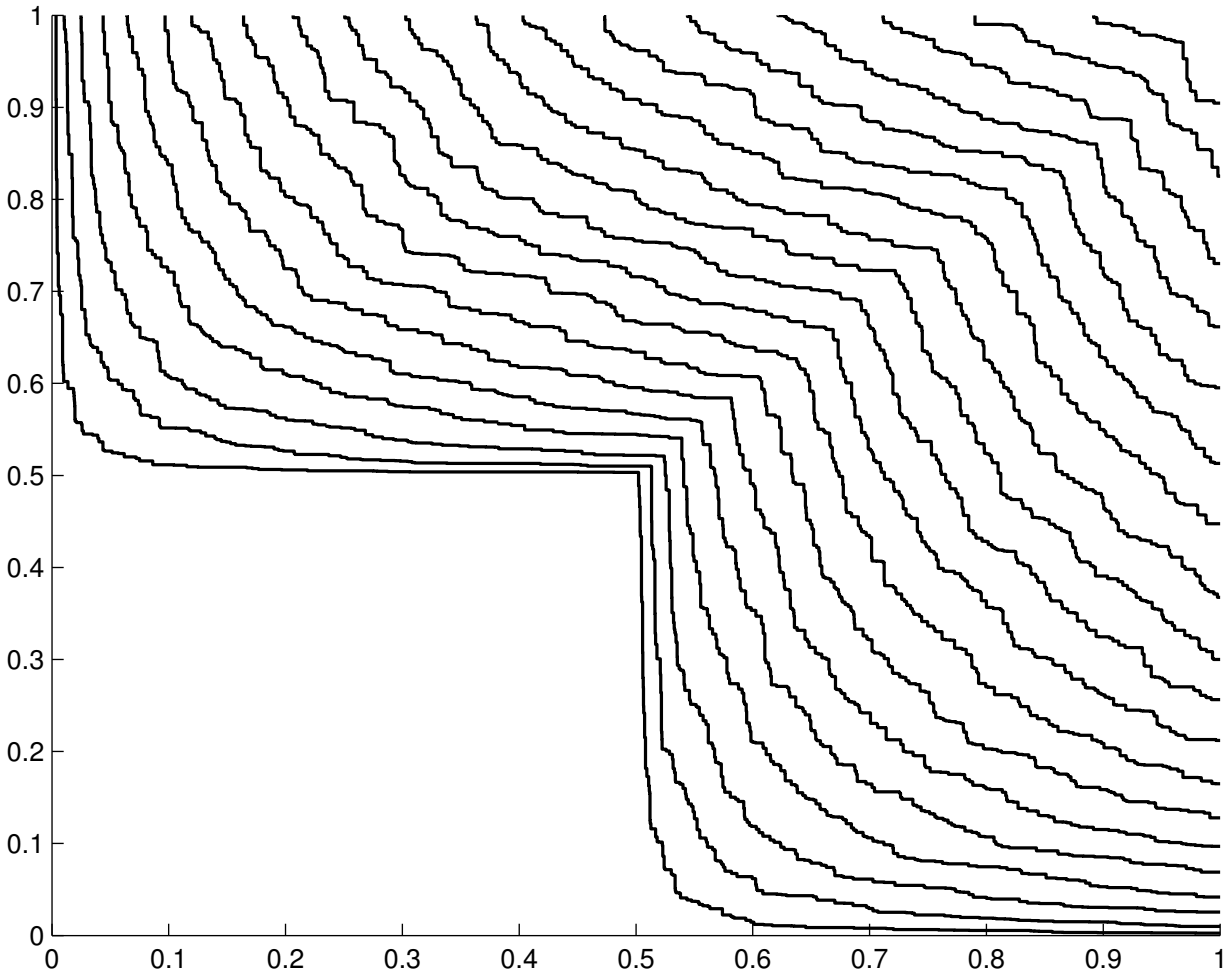

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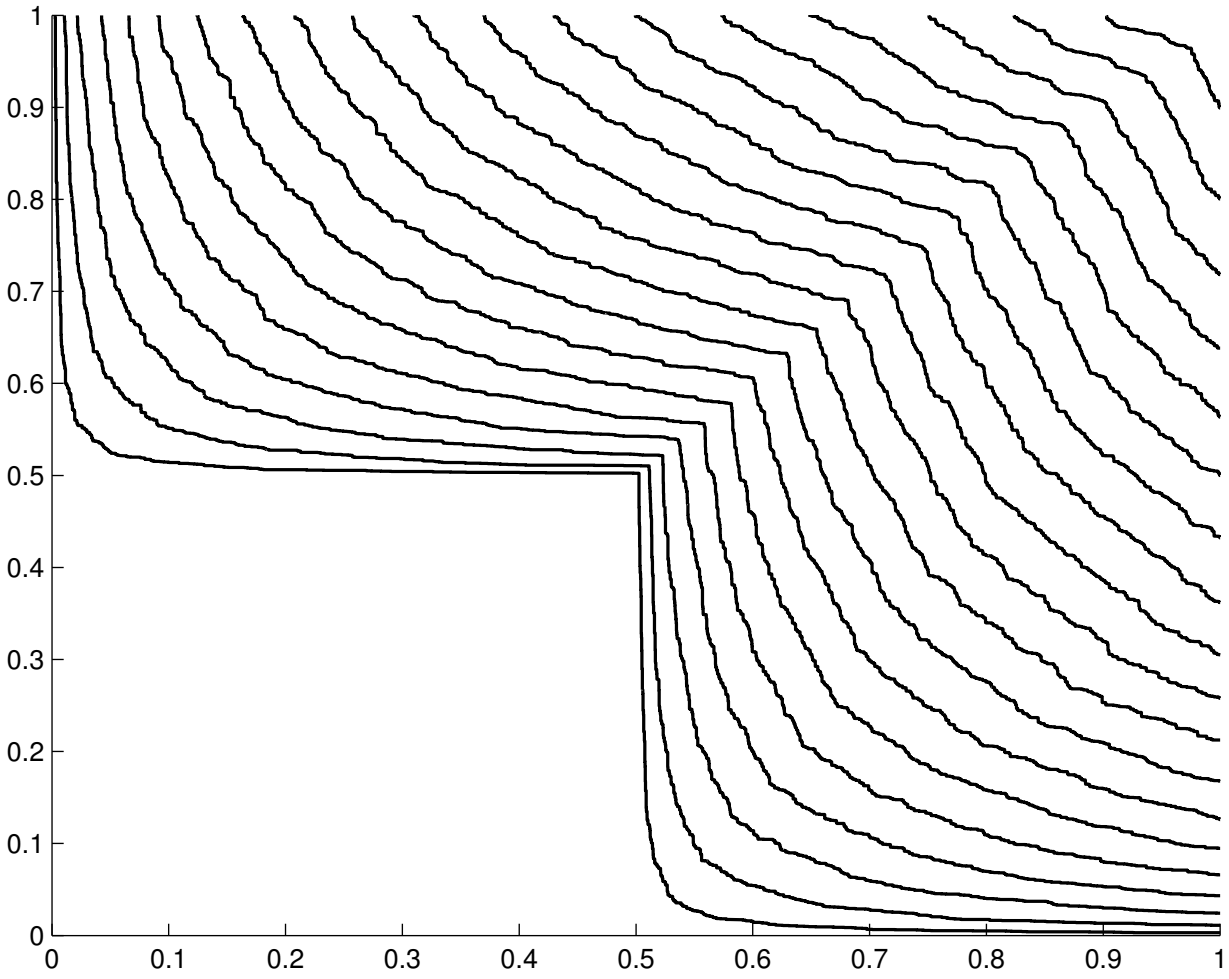

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Question

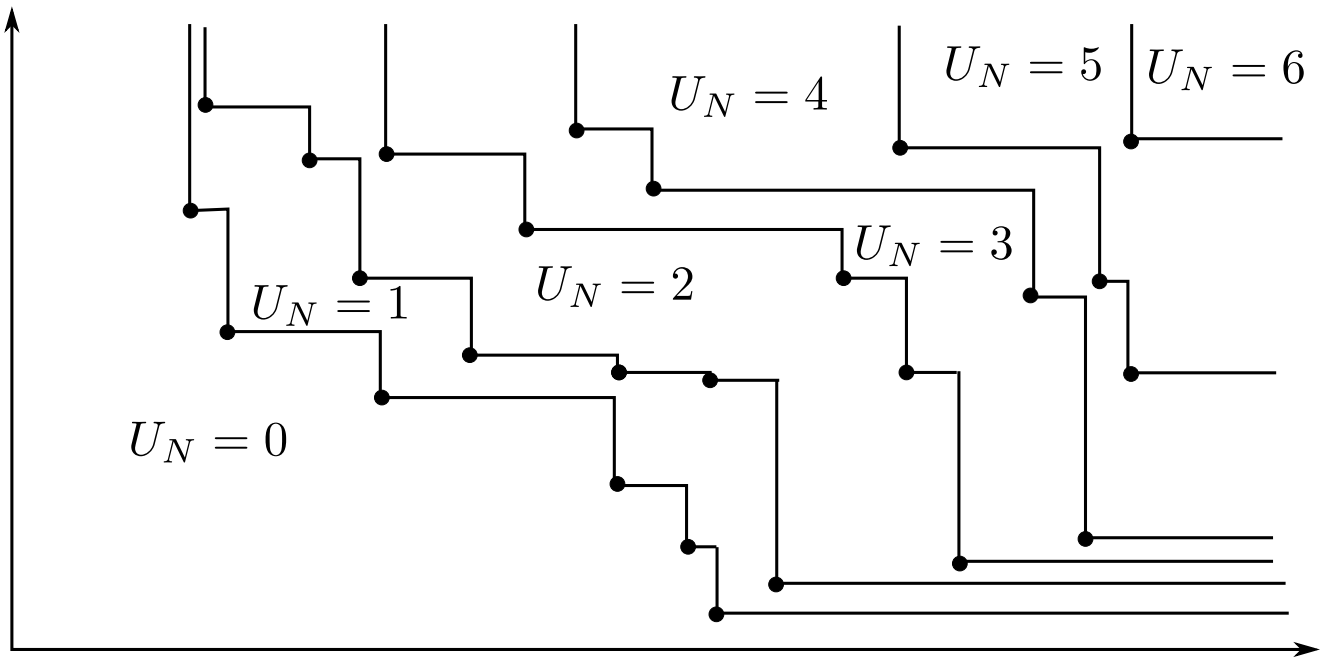
Can we characterize the asymptotic shapes of the Pareto fronts?

Outline

A PDE continuum limit for non-dominated sorting

Let $X_1, \ldots, X_N$ be *i.i.d.* random variables in $[0, \infty)^n$ with continuous density f.

Let $U_N : \mathbb{R}^n \to \mathbb{N}_0$ be the function that 'counts' the layers $\mathcal{F}_1, \mathcal{F}_2, \ldots$



Continuum limit

Theorem (Calder, Esedoḡlu, Hero, 2014)

With probability one

$$N^{-\frac{1}{d}} U_N \longrightarrow u \quad \textit{locally uniformly on } [0,\infty)^n,$$

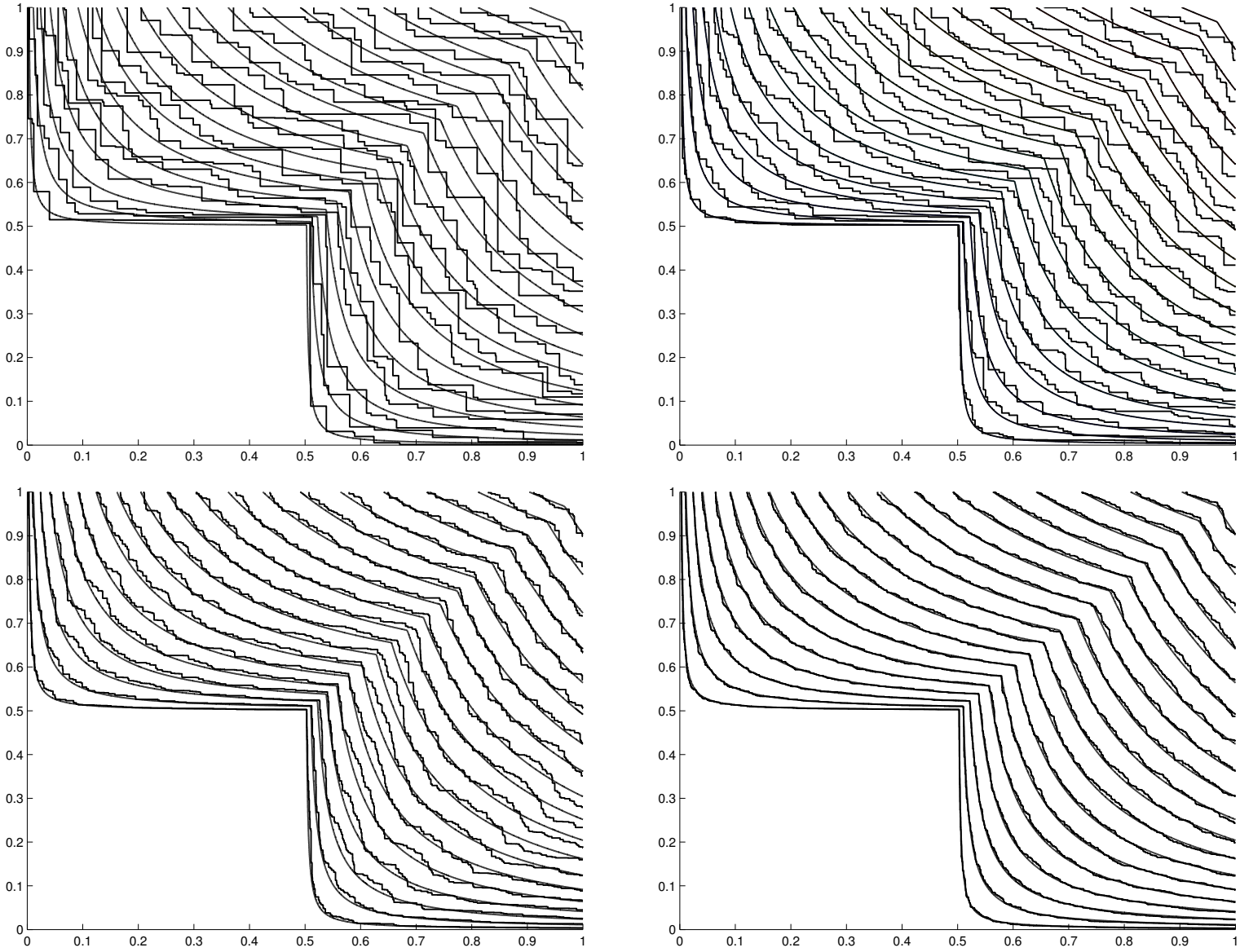
where $u \in C^{0,\frac{1}{n}}([0,\infty)^n)$ *is the unique nondecreasing viscosity solution of*

$$\boxed{\text{(P1)} \begin{cases} u_{x_1} \cdots u_{x_n} = c_n f & \textit{in } \mathbb{R}^n_+ := (0,\infty)^n \\ u = 0 & \textit{on } \partial\mathbb{R}^n_+. \end{cases}}$$

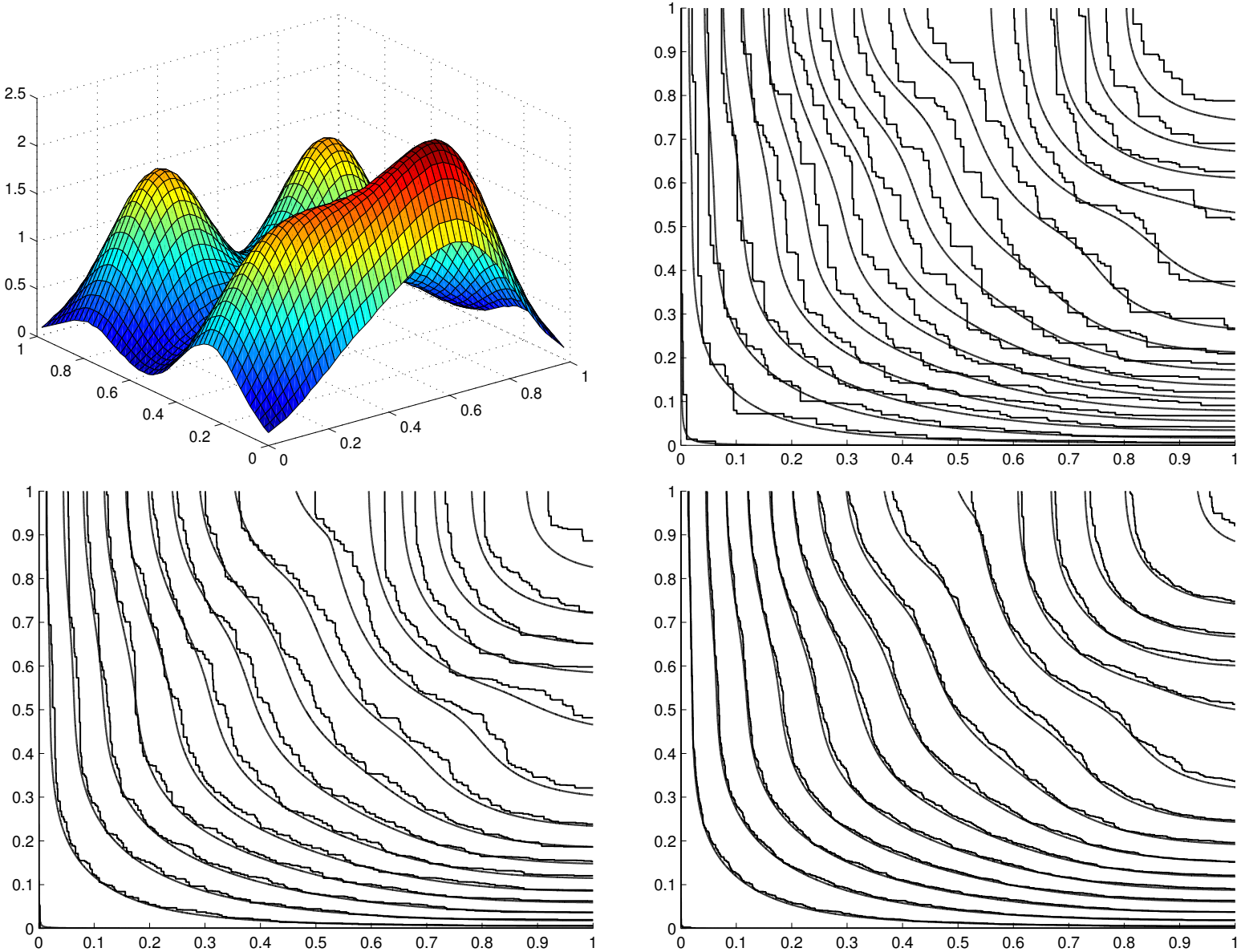
Calder, J., Esedoḡlu, S., and Hero, A. O. (2014). A Hamilton-Jacobi equation for the continuum limit of non-dominated sorting. *SIAM Journal on Mathematical Analysis*, 46(1):603–638.

Calder, J. (2015). A direct verification argument for the Hamilton-Jacobi equation continuum limit of nondominated sorting. *arXiv preprint:1508.01565*

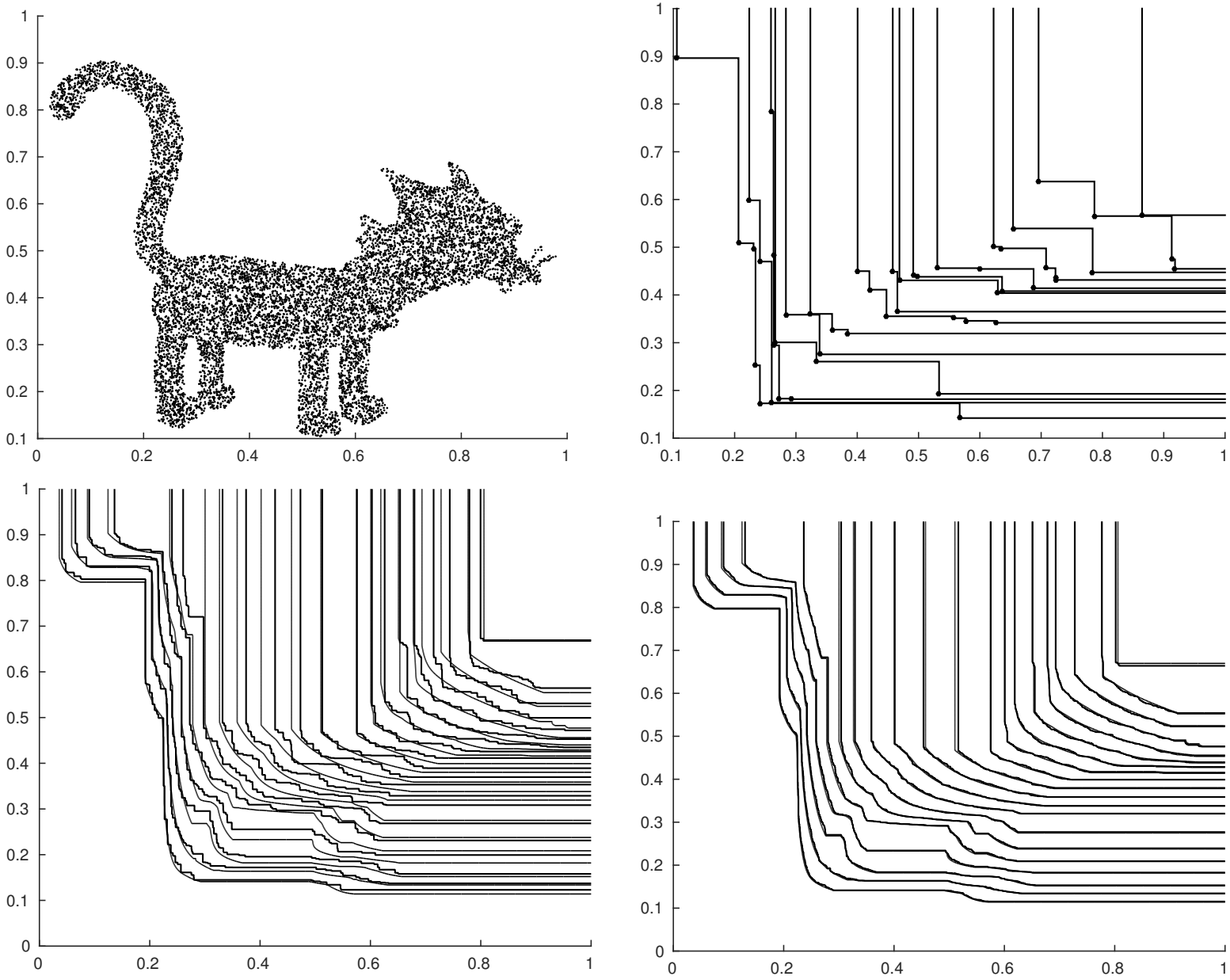
Demo: $f = 1 - \chi_{[0,0.5]^2}$



Demo: Multimodal f



Demo: A Cat



Fast approximate sorting

Algorithm (PDE-based Ranking)

1. *Select k points from $X_1, \ldots, X_N$ at random. Call them $Y_1, \ldots, Y_k$.*
2. *Select a grid spacing h for solving the PDE (P1) and estimate f with a histogram aligned to the grid $[0,1]_h^n$, i.e.,*

$$\hat{f}(x) = \frac{1}{kh^n} \cdot \#\Big\{ Y_i \,:\, x \leqq Y_i \leqq x + h(1, \ldots, 1) \Big\} \text{ for } x \in [0,1]_h^n.$$

3. *Compute the numerical solution $\hat{U}_h$ on $[0,1]_h^n$.*
4. *Evaluate $\hat{U}_h(X_i)$ for $i = 1, \ldots, N$ via interpolation.*

Notes:

- Total complexity is $O(k + h^{-n} + N)$.
- If we fix k, h and n, independent of N, then Steps 1-3 have $O(1)$ complexity.

Calder, J., Esedoḡlu, S., and Hero, A. O. (2015). A PDE-based approach to nondominated sorting. *SIAM Journal on Numerical Analysis*, 53(1):82–104.

Outline

Numerics

How do we numerically solve

$$\text{(P1)} \begin{cases} u_{x_1} \cdots u_{x_n} = f & \text{in} \quad \mathbb{R}^n_+ \\ u = 0 & \text{on} \quad \partial\mathbb{R}^n_+ \end{cases}$$

efficiently and accurately (in dimensions $n = 2, 3, 4$)?

Recall: Viscosity solution

Consider the Hamilton-Jacobi equation

$$H(x, u, Du) = 0 \quad \text{on } \mathcal{O} \subseteq \mathbb{R}^n, \tag{1}$$

where $H : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

A continuous function $u : \mathcal{O} \to \mathbb{R}$ is a viscosity solution of (1) if

1. **Subsolution:** For every $x \in \mathcal{O}$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local maximum at x with respect to $\mathcal{O}$

$$H(x, u(x), D\varphi(x)) \leq 0.$$

2. **Supersolution:** For every $x \in \mathcal{O}$ and $\varphi \in C^\infty(\mathbb{R}^n)$ such that $u - \varphi$ has a local minimum at x with respect to $\mathcal{O}$

$$H(x, u(x), D\varphi(x)) \geq 0.$$

Bounded domain

We first pose the PDE on a bounded domain

$$\text{(P1)} \begin{cases} u_{x_1} \cdots u_{x_n} = f & \text{in} \quad (0,1]^n \\ u = 0 & \text{on} \quad \Gamma \end{cases}$$

where $\Gamma = [0,1]^n \setminus (0,1]^n$.

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We actually need to slightly modify the PDE:

$$\text{(P1')} \begin{cases} (u_{x_1})_+ \cdots (u_{x_n})_+ = f & \text{in } (0,1]^n \\ u = 0 & \text{on } \Gamma, \end{cases}$$

where $t_+ = \max(t, 0)$.

Upwind finite difference scheme for (P1)

The upwind scheme corresponds to using backward differences:

$$\text{(S1)} \begin{cases} D_1^- u_h(x) \cdots D_n^- u_h(x) = f(x) & \text{for } x \in (0,1]_h^n \\ u_h(x) = 0 & \text{for } x \in \Gamma_h, \end{cases}$$

where $\Omega_h := \Omega \cap h\mathbb{Z}^n$ and

$$D_i^{\pm} u_h(x) = \pm \frac{u_h(x \pm he_i) - u_h(x)}{h}.$$

Upwind finite difference scheme for (P1)

The upwind scheme corresponds to using backward differences:

$$\text{(S1)} \begin{cases} D_1^- u_h(x) \cdots D_n^- u_h(x) = f(x) & \text{for } x \in (0,1]_h^n \\ u_h(x) = 0 & \text{for } x \in \Gamma_h, \end{cases}$$

where $\Omega_h := \Omega \cap h\mathbb{Z}^n$ and

$$D_i^{\pm} u_h(x) = \pm \frac{u_h(x \pm he_i) - u_h(x)}{h}.$$

The scheme (S) can be solved in a single pass (similar to fast sweeping or marching), and in dimension $n = 2$ we have the closed form expression

$$u_h(x) = \frac{u_h(x - he_1) + u_h(x - he_2)}{2} + \frac{1}{2}\sqrt{(u_h(x - he_1) - u_h(x - he_2))^2 + 4h^2 f(x)^2}.$$

However, accuracy is poor $O(h^{\frac{1}{n}})$.

Calder, J., Esedoḡlu, S., and Hero, A. O. (2015). A PDE-based approach to nondominated sorting. *SIAM Journal on Numerical Analysis*, 53(1):82–104.

Accuracy of (S1)

For $f \equiv 1$, $u(x) = n(x_1 \cdots x_n)^{\frac{1}{n}}$. Set $\varphi(x) = Cn(x_1 \cdots x_n)^{\frac{1}{n}}$ and by concavity

$$D_i^- \varphi(x) \geq \varphi_{x_i}(x) = C(x_1 \cdots x_n)^{\frac{1}{n}} x_i^{-1}.$$

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Therefore, for any $x \in (0,1]_h^n$ such that $x_i = h$ for some i we have

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Therefore

$$u(x) - u_h(x) \geq \left(n - n^{1-\frac{1}{n}} \right) h^{\frac{1}{n}}.$$

Towards a new scheme

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$$v_{x_1} \cdots v_{x_n} = v^{n-1} \underbrace{u_{x_1} \cdots u_{x_n}}_{f} = v^{n-1} f.$$

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We find that v is a viscosity solution of

$$\boxed{\text{(P2)} \begin{cases} v_{x_1} \cdots v_{x_n} = v^{n-1} f & \text{in } (0,1]^n \\ v = 0 & \text{on } \Gamma \end{cases}}$$

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Since $f \geq 0$, (P2) has a zeroth order term of the wrong sign for a comparison principle. The method of vanishing viscosity takes the form

$$v^\varepsilon_{x_1} \cdots v^\varepsilon_{x_n} - \varepsilon \Delta v^\varepsilon = (v^\varepsilon)^{n-1} f.$$

Nonuniqueness for (P2)

$$\text{(P2)} \begin{cases} v_{x_1} \cdots v_{x_n} = v^{n-1} f & \text{in } (0,1]^n \\ v = 0 & \text{on } \Gamma \end{cases}$$

When $f \equiv 1$, $v(x) = u(x)^n / n^n = x_1 \cdots x_n$, but

$$v^y(x) := (x_1 - y_1)_+ \cdots (x_n - y_n)_+$$

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Lemma

Assume f is continuous on $[0,1]^n$. Then $v := u^n / n^n$ is the unique maximal viscosity solution of (P2).

Numerical scheme for (P2)

An upwind scheme for (P2) uses backward difference quotients

$$\text{(S2)} \begin{cases} D_1^- v_h(x) \cdots D_n^- v_h(x) = v_h^{n-1}(x) f(x) & \text{for } x \in (0,1]_h^n \\ v_h(x) = 0 & \text{for } x \in \Gamma_h, \end{cases}$$

We define v_h by taking the largest solution of (S2) at each $x \in (0,1]_h^n$.

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The scheme can be solved efficiently in a single pass, and in dimension $n = 2$ the scheme can be solved in closed form

$$v_h(x) = \frac{A + h^2 f(x)}{2} + \frac{1}{2}\sqrt{B^2 + 2h^2 f(x) A + h^4 f(x)^2},$$

where

$$A = v_h(x - he_1) + v_h(x - he_2) \quad \text{and} \quad B = v_h(x - he_1) - v_h(x - he_2).$$

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Notice when $f \equiv 1$, $v_h(x) = x_1 \cdots x_n$ is the exact solution of (P2).

Rate of convergence for (S2)

$$\text{(S2)} \begin{cases} D_1^- v_h(x) \cdots D_n^- v_h(x) = v_h^{n-1}(x) f(x) & \text{for } x \in (0,1]_h^n \\ v_h(x) = 0 & \text{for } x \in \Gamma_h, \end{cases}$$

The scheme (S2) has formal accuracy of $O(h)$ and we have

Theorem

Suppose $f \in C^{0,1}([0,1]^n)$ *and* $f > 0$. *Then*

$$|n^n v_h - u^n| \leq C\sqrt{h} \quad \textit{in } [0,1]_h^n, \tag{2}$$

and

$$|n v_h^{\frac{1}{n}} - u| \leq C\delta^{1-n}\sqrt{h} \quad \textit{in } [\delta,1]_h^n, \tag{3}$$

where $\delta > 0$ *and* $C = C\big(n, [f]_{1;[0,1]^n}, \min_{[0,1]^n} f\big)$.

Calder, J. (2015). Numerical schemes and rates of convergence for the Hamilton-Jacobi equation continuum limit of nondominated sorting. *arXiv preprint:1508.01557*.

One-sided rate for (S1)

Recall u_h satisfies

$$D_1^- u_h(x) \cdots D_n^- u_h(x) = f(x) \quad \text{for } x \in (0,1]_h^n.$$

One-sided rate for (S1)

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Let $\psi(x) = u_h^n / n^n$. By convexity and monotoncity

$$D_i^- \psi(x) = \frac{u_h(x)^n - u_h(x - he_i)^n}{n^n h} \leq \frac{u_h(x)^{n-1}}{n^{n-1}} D_i^- u_h(x).$$

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Corollary

Suppose $f \in C^{0,1}([0,1]^n)$ and $f > 0$. Then

$$u_h^n - u^n \leq C\sqrt{h}.$$

Regularity

To prove the $O(\sqrt{h})$ convergence rate, we need a Lipschitz regularity result for (P2) or (S2). Textbook regularity results are based on ($p = Du$)

$$\text{(Coercivity)} \qquad \lim_{|p|\to\infty} H(x,p) = \infty \quad \text{uniformly in } x,$$

or

$$\text{(Zeroth order term)} \qquad u + H(x, Du) = 0.$$

Refer to [Bardi and Dolcetta, 1997]

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Refer to [Bardi and Dolcetta, 1997]

The Hamiltonian for (P2) is $(z = u, p = Du)$

$$H(x,z,p) = p_1 \cdots p_n - z^{n-1} f(x)$$

which satisfies neither.

Regularity for $n = 2$

Differentiate both sides of (P2) $\frac{\partial}{\partial x}(v_x v_y) = \frac{\partial}{\partial x}(vf)$ to find that

$$v_{xx} v_y + v_x v_{xy} = v_x f + v f_x.$$

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Set $\varphi = v_x$ and rearrange

$$\varphi_x v_y + (\varphi_y - f)\varphi = v f_x \leq \|f\|_{L^\infty} [f]_{1;[0,1]^n} .$$

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Lemma

Let $f \in C^{0,1}([0,1]^2)$ be nonnegative. Then there exists $C > 0$ such that

$$[v]_{1;[0,1]^2} \leq C\|f\|_{C^{0,1}([0,1]^2)}. \tag{4}$$

Sketch of proof

Differentiate numerical scheme $D_x^-(D_x^- v D_y^- v) = D_x^-(vf)$ $\qquad (\mathbf{x} = (x, y))$

$$D_x^- D_x^- v(\mathbf{x}) D_y^- v(\mathbf{x} - he_x) + D_x^- v(\mathbf{x}) D_x^- D_y^- v(\mathbf{x}) = v(\mathbf{x}) D_x^- f(\mathbf{x}) + f(\mathbf{x} - he_x) D_x^- v(\mathbf{x}).$$

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Set $\overline{\varphi}(x, y) = C(1 + y)$ and compute

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We can select

$$C = \|f\|_{L^\infty} + \sqrt{\|f\|_{L^\infty} [f]_1}$$

and we have

$$\boxed{\Big(D_y^- \overline{\varphi}(\mathbf{x}) - f(\mathbf{x} - he_x)\Big)\overline{\varphi}(\mathbf{x}) > \|f\|_{L^\infty} [f]_1}.$$

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Hence

$$\boxed{(D_y^- \varphi(\mathbf{x}) - f(\mathbf{x} - he_x))\varphi(\mathbf{x}) \geq (D_y^- \overline{\varphi}(\mathbf{x}) - f(\mathbf{x} - he_x))\overline{\varphi}(\mathbf{x}).}$$

Sketch of proof

We claim $\varphi \leq \overline{\varphi} := C(1+y)$. Since $v(x,y) \leq \|f\|_{L^\infty} xy$, $\varphi \leq \overline{\varphi}$ on $\Gamma = [0,1]^2 \setminus (0,1]^2$. Assume to the contrary that $\varphi(\mathbf{x}) > \overline{\varphi}(\mathbf{x})$ for some $\mathbf{x} \in (0,1]^2$. Then there exists $\mathbf{x}$ such that

$$\varphi(\mathbf{x}) > \overline{\varphi}(\mathbf{x}),\ \varphi(\mathbf{x} - he_x) \leq \overline{\varphi}(\mathbf{x} - he_x), \quad \text{and} \quad \varphi(\mathbf{x} - he_y) \leq \overline{\varphi}(\mathbf{x} - he_y).$$

Then

$$D_x^- \varphi(\mathbf{x}) \geq D_x^- \overline{\varphi}(\mathbf{x}) = 0 \quad \text{and} \quad D_y^- \varphi(\mathbf{x}) \geq D_y^- \overline{\varphi}(\mathbf{x}) \geq \|f\|_{L^\infty}.$$

Hence

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Recalling

$$\boxed{\underbrace{D_x^- \varphi(\mathbf{x}) D_y^- v(\mathbf{x} - he_x)}_{\geq 0} + \Big(D_y^- \varphi(\mathbf{x}) - f(\mathbf{x} - he_x)\Big)\varphi(\mathbf{x}) = v D_x^- f \leq \|f\|_{L^\infty} [f]_1.}$$

$$\boxed{\Big(D_y^- \overline{\varphi}(\mathbf{x}) - f(\mathbf{x} - he_x)\Big)\overline{\varphi}(\mathbf{x}) > \|f\|_{L^\infty} [f]_1,}$$

we have a contradiction.

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we have a contradiction. Thus

$$D_x^- v \le 2C = 2\left(\|f\|_{L^\infty} + \sqrt{\|f\|_{L^\infty} [f]_1}\right) \le C\|f\|_{C^{0,1}}. \quad \square$$

Regularity for $n = 3$

Differentiate both sides of (P2) $\frac{\partial}{\partial x}(v_x v_y v_z) = \frac{\partial}{\partial x}(v^2 f)$ to find that

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Set $\varphi = v_x$ and rearrange

$$F(\varphi) := \varphi_x v_y v_z + (v_z \varphi_y + v_y \varphi_z - 2vf)\varphi = v^2 f_x \leq \|f\|_{L^\infty}^2 [f]_1.$$

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Perhaps we should be looking for a supersolution of the form

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At a maximum of φ, $\varphi_x = \varphi_y = \varphi_z = 0$ so

$$-2vf\varphi \leq \|f\|_{L^\infty}^2 [f]_1.$$

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Back to the drawing board!

Another PDE

Recall for $f \equiv 1$, $u(x) = n(x_1 \cdots x_n)^{\frac{1}{n}}$. For general f, we are tempted to make the ansatz

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Proceeding formally

$$u_{x_i} = (x_1 \cdots x_n)^{\frac{1}{n}} \frac{w}{x_i} + n(x_1 \cdots x_n)^{\frac{1}{n}} w_{x_i} = \frac{1}{x_i}(x_1 \cdots x_n)^{\frac{1}{n}}(w + nx_i w_{x_i}).$$

Therefore

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Therefore

$$f = u_{x_1} \cdots u_{x_n} = \prod_{i=1}^{n}(w + nx_i w_{x_i}).$$

We can show that w is a bounded viscosity solution of

$$\boxed{\text{(P3)} \quad \prod_{i=1}^{n}(w + nx_i w_{x_i}) = f \quad \text{in } (0,1]^n.}$$

Notice:

- Zeroth order term has correct sign! (since $u_{x_i} \geq 0$)
- $v = u^n/n^n = x_1 \cdots x_n w^n$.

Regularity for (P3)

Let us rewrite (P3) as

$$\sum_{j=1}^{n} \log(w + n x_j w_{x_j}) = \log(f) \quad \text{in } (0,1]^n.$$

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$$\sum_{j=1}^{n} \frac{w_{x_i} + n\delta_{i,j}\, w_{x_j} + nx_j\, w_{x_i x_j}}{w + nx_j\, w_{x_j}} = \frac{f_{x_i}}{f}.$$

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By the inequality of arithmetic and geometric means

$$\sum_{j=1}^{n} \frac{1 + n\delta_{i,j}}{w + nx_j\, w_{x_j}} \geq \sum_{j=1}^{n} \frac{1}{w + nx_j\, w_{x_j}} \geq n \prod_{j=1}^{n} (w + nx_j\, w_{x_j})^{-\frac{1}{n}} = nf^{-\frac{1}{n}}.$$

Regularity for (P3)

Hence, at a positive maximum of w_{x_i} we have

$$w_{x_i} \leq \frac{1}{n} f^{\frac{1}{n}-1} f_{x_i} = (f^{\frac{1}{n}})_{x_i} \leq [f^{\frac{1}{n}}]_{1;[0,1]^n}.$$

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Problem: We have no boundary condition for w, how do we know w_{x_i} attains a maximum or minimum?

Recall

$$\text{(P3)} \quad \prod_{i=1}^{n} (w + n x_i w_{x_i}) = f \quad \text{ in } (0,1]^n.$$

When $f \equiv 1$, the solution of (P3) that we care about is $w \equiv 1$. For any $C > 0$

$$w(x) = \prod_{i=1}^{n} (1 + C x_i^{-1})^{\frac{1}{n}}$$

is an unbounded solution of (P3).

Extension property for (P3)

$$\text{(P3)} \quad \prod_{i=1}^{n}(w + nx_i w_{x_i}) = f \quad \text{in } (0,1]^n.$$

Lemma

Let $f \geq 0$ be continuous on $[0,1]^n$ and let u be the solution of (P1). Then

$$w(x) = \frac{u(x)}{n(x_1 \cdots x_n)^{\frac{1}{n}}}$$

can be extended to a continuous function $w \in C((-\infty,1]^n)$ satisfying

$$w(x) = w(x_+) \quad \text{for all } x \in (-\infty,1]^n,$$

where $x_+ = (\max(x_1,0),\ldots,\max(x_n,0))$. Furthermore, if we extend f by setting $f(x) := f(x_+)$ then w is a viscosity solution of

$$\prod_{i=1}^{n}(w + nx_i w_{x_i}) = f \quad \text{in} \quad (-\infty,1]^n.$$

Regularity for (P3)

The extension property allows us to complete the regularity result.

Theorem

Let f be a nonnegative function for which $f^{\frac{1}{n}} \in C^{0,1}([0,1]^n)$, and let w be the maximal bounded viscosity solution of (P3). Then $w \in C^{0,1}([0,1]^n)$ and

$$[w]_{1;[0,1]^n} \leq \sqrt{n}\,[f^{\frac{1}{n}}]_{1;[0,1]^n}. \tag{5}$$

Theorem is sharp: When $f(x) = (x_1 \cdots x_n)^{n-1}$, $w(x) = \frac{1}{n}(x_1 \cdots x_n)^{1-\frac{1}{n}} \notin C^{0,1}([0,1]^n)$.

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Since $v = x_1 \cdots x_n w^n$ we have the following corollary.

Corollary

Let v be the maximal viscosity solution of (P2). Then $v \in C^{0,1}([0,1]^n)$ and

$$[v]_{1;[0,1]^n} \leq C\|f^{\frac{n-1}{n}}\|_{L^\infty([0,1]^n)}\|f^{\frac{1}{n}}\|_{C^{0,1}([0,1]^n)}. \tag{6}$$

Numerical scheme for (P3)

An upwind scheme for (P3) uses backward difference quotients

$$\text{(S3)} \qquad \prod_{i=1}^{n} \left(w_h(x) + n x_i D_i^- w_h(x) \right) = f(x) \quad \text{for } x \in [0,1]_h^n.$$

We define w_h by taking the largest solution of (P2) at each $x \in [0,1]_h^n$.
Notice

- $w_h(0) = f(0)^{\frac{1}{n}}$.

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Notice

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The scheme can be solved efficiently in a single pass, and in dimension $n = 2$ the scheme can be solved in closed form

$$w_h(x) = C + \sqrt{D^2 + (2x_1 + h)(2x_2 + h)h^2 f(x)},$$

where

$$C = x_1(2x_2 + h)w_h(x - he_1) + x_2(2x_1 + h)w_h(x - he_2),$$

and

$$D = x_1(2x_2 + h)w_h(x - he_1) - x_2(2x_1 + h)w_h(x - he_2).$$

Convergence rate

Since (P3) has a zeroth order term of the correct sign, and w and w_h are Lipschitz continuous on $[0,1]^n$ we can prove

Theorem

Suppose that $f \in C^{0,1}([0,1]^n)$ *and* $f > 0$. *Then*

$$|w(x) - w_h(x)| \leq C\sqrt{h} \quad \textit{for all } x \in [0,1]_h^n, \tag{7}$$

where $C = C\big(n, [f]_{1;[0,1]^n}, \min_{[0,1]^n} f\big)$.

Directed Hamiltonians

All of the PDE in this talk have a common useful property.

$$H(x, u(x), Du(x)) = 0.$$

Definition

We say that $H(x, z, p)$ is directed if

$$H(x, z, p) \leq H(x, z, q) \text{ whenever } p_i \leq q_i \text{ for all } i.$$

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Recall:

$$\begin{aligned}
&\text{(P1)} \quad H(x, z, p) = (p_1)_+ \cdots (p_n)_+ - f(x) \\
&\text{(P2)} \quad H(x, z, p) = (p_1)_+ \cdots (p_n)_+ - z^{n-1} f(x) \\
&\text{(P3)} \quad H(x, z, p) = \prod_{i=1}^{n} (z + n x_i p_i)_+ - f(x)
\end{aligned}$$

Monotone (upwind) scheme for directed H

A monotone scheme for a directed Hamiltonian H is always to use backward differences

$$H(x, u(x), D^- u(x)) = 0,$$

where

$$D^- u(x) = (D_1^- u(x), \ldots, D_n^- u(x)).$$

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Indeed, if $u \geq v$ and $u(x) = v(x)$ then $D_i^- u(x) \leq D_i^- v(x)$ for all i and

$$H(x, u(x), D^- u(x)) \leq H(x, v(x), D^- v(x)),$$

which is exactly monotonicity of the scheme.

Convergence rate

$$\text{(S)} \begin{cases} u_h + H(D^- u_h) = f & \text{for } x \in (0,1]_h^n \\ u_h = g & \text{for } x \in \Gamma_h, \end{cases}$$

$$\text{(H)} \begin{cases} u + H(Du) = f & \text{in } (0,1]^n \\ u = g & \text{on } \Gamma, \end{cases}$$

Theorem ([Crandall and Lions, 1984, Souganidis, 1985])

Assume $H \in C^{0,1}(\mathbb{R}^n)$ is directed and $f, g \in C^{0,1}([0,1]^n)$. Let u_h be a solution of (S) and let u be a viscosity solution of (H). If $u \in C^{0,1}([0,1]^n)$ and

$$\sup_{h>0} \sup_{x \neq y} \frac{|u^h(x) - u^h(y)|}{|x-y|} < \infty,$$

then

$$|u_h - u| \leq C\sqrt{h}.$$

Sketch of proof that $u_h \leq u + C\sqrt{h}$

For $\alpha > 0$ set

$$\Phi(x, y) = u_h(x) - u(y) - \frac{\alpha}{2}|x - y|^2,$$

and let $(x_\alpha, y_\alpha) \in (0, 1]_h^n \times (0, 1]^n$ such that $\Phi(x_\alpha, y_\alpha) = \max_{[0,1]_h^n \times [0,1]^n} \Phi$.

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Set

$$\varphi(x) = \frac{\alpha}{2}|x - y_\alpha|^2 - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 + u_h(x_\alpha).$$

Then $\varphi(x_\alpha) = u_h(x_\alpha)$ and $u_h(x) \leq \varphi(x)$ for all $x \in [0, 1]_h^n$.

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$$f(x_\alpha) = u_h(x_\alpha) + H(D^- u_h(x_\alpha)) \geq u_h(x_\alpha) + H(D^- \varphi(x_\alpha)).$$

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Since $H \in C^{0,1}(\mathbb{R}^n)$ and $D^- \varphi(x_\alpha) = p_\alpha - \frac{\alpha h}{2}(1, \ldots, 1)$ where $p_\alpha = \alpha(x_\alpha - y_\alpha)$

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$$u_h(x_\alpha) + H(p_\alpha) \leq f(x_\alpha) + C\alpha h.$$

Let $\psi(y) = \frac{\alpha}{2}|x_\alpha - y|^2$. Since $u + \psi$ has a minimum at y_α and $-D\psi(y_\alpha) = p_\alpha$

$$u(y_\alpha) + H(p_\alpha) \geq f(y_\alpha).$$

Sketch of proof that $u_h \leq u + C\sqrt{h}$

For $\alpha > 0$ set

$$\Phi(x,y) = u_h(x) - u(y) - \frac{\alpha}{2}|x-y|^2,$$

and let $(x_\alpha, y_\alpha) \in (0,1]_h^n \times (0,1]^n$ such that $\Phi(x_\alpha, y_\alpha) = \max_{[0,1]_h^n \times [0,1]^n} \Phi$.

Set

$$\varphi(x) = \frac{\alpha}{2}|x-y_\alpha|^2 - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2 + u_h(x_\alpha).$$

Then $\varphi(x_\alpha) = u_h(x_\alpha)$ and $u_h(x) \leq \varphi(x)$ for all $x \in [0,1]_h^n$. Since H is directed

$$f(x_\alpha) = u_h(x_\alpha) + H(D^- u_h(x_\alpha)) \geq u_h(x_\alpha) + H(D^- \varphi(x_\alpha)).$$

Since $H \in C^{0,1}(\mathbb{R}^n)$ and $D^-\varphi(x_\alpha) = p_\alpha - \frac{\alpha h}{2}(1,\ldots,1)$ where $p_\alpha = \alpha(x_\alpha - y_\alpha)$

$$u_h(x_\alpha) + H(p_\alpha) \leq f(x_\alpha) + C\alpha h.$$

Let $\psi(y) = \frac{\alpha}{2}|x_\alpha - y|^2$. Since $u + \psi$ has a minimum at y_α and $-D\psi(y_\alpha) = p_\alpha$

$$u(y_\alpha) + H(p_\alpha) \geq f(y_\alpha).$$

Hence: $\boxed{u_h(x_\alpha) - u(y_\alpha) \leq f(x_\alpha) - f(y_\alpha) + C\alpha h.}$

Sketch of proof that $u_h \leq u + C\sqrt{h}$

Since $\Phi(x,y) = u_h(x) - u(y) - \frac{\alpha}{2}|x-y|^2$

$$\max_{[0,1]_h^n}(u_h - u) \leq \Phi(x_\alpha, y_\alpha) \leq u_h(x_\alpha) - u(y_\alpha) \leq f(x_\alpha) - f(y_\alpha) + C\alpha h.$$

Sketch of proof that $u_h \leq u + C\sqrt{h}$

Since $\Phi(x,y) = u_h(x) - u(y) - \frac{\alpha}{2}|x-y|^2$

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Since $\Phi(x_\alpha, x_\alpha) \leq \Phi(x_\alpha, y_\alpha)$ and $u \in C^{0,1}([0,1]^n)$

$$u_h(x_\alpha) - u(x_\alpha) \leq u_h(x_\alpha) - u(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2$$

or

$$\frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \leq u(x_\alpha) - u(y_\alpha) \leq C|x_\alpha - y_\alpha|.$$

Hence $|x_\alpha - y_\alpha| \leq C/\alpha$.

Sketch of proof that $u_h \leq u + C\sqrt{h}$

Since $\Phi(x,y) = u_h(x) - u(y) - \frac{\alpha}{2}|x-y|^2$

$$\max_{[0,1]_h^n}(u_h - u) \leq \Phi(x_\alpha, y_\alpha) \leq u_h(x_\alpha) - u(y_\alpha) \leq f(x_\alpha) - f(y_\alpha) + C\alpha h.$$

Since $\Phi(x_\alpha, x_\alpha) \leq \Phi(x_\alpha, y_\alpha)$ and $u \in C^{0,1}([0,1]^n)$

$$u_h(x_\alpha) - u(x_\alpha) \leq u_h(x_\alpha) - u(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2$$

or

$$\frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \leq u(x_\alpha) - u(y_\alpha) \leq C|x_\alpha - y_\alpha|.$$

Hence $|x_\alpha - y_\alpha| \leq C/\alpha$. Therefore

$$\max_{[0,1]_h^n}(u_h - u) \leq C\left(\frac{1}{\alpha} + \alpha h\right).$$

Sketch of proof that $u_h \leq u + C\sqrt{h}$

Since $\Phi(x,y) = u_h(x) - u(y) - \frac{\alpha}{2}|x-y|^2$

$$\max_{[0,1]_h^n}(u_h - u) \leq \Phi(x_\alpha, y_\alpha) \leq u_h(x_\alpha) - u(y_\alpha) \leq f(x_\alpha) - f(y_\alpha) + C\alpha h.$$

Since $\Phi(x_\alpha, x_\alpha) \leq \Phi(x_\alpha, y_\alpha)$ and $u \in C^{0,1}([0,1]^n)$

$$u_h(x_\alpha) - u(x_\alpha) \leq u_h(x_\alpha) - u(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2$$

or

$$\frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \leq u(x_\alpha) - u(y_\alpha) \leq C|x_\alpha - y_\alpha|.$$

Hence $|x_\alpha - y_\alpha| \leq C/\alpha$. Therefore

$$\max_{[0,1]_h^n}(u_h - u) \leq C\left(\frac{1}{\alpha} + \alpha h\right).$$

The best possible choice for α is $\alpha = 1/\sqrt{h}$ which yields

$$u_h - u \leq C\sqrt{h}. \quad \square$$

Convergence rate for (S3)

$$\boxed{\text{(P3)} \quad \prod_{i=1}^{n}(w + nx_i w_{x_i}) = f \quad \text{in } (0,1]^n.}$$

The proof that $|w - w_h| \leq C\sqrt{h}$ is similar, with the extension property replacing the boundary condition.

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Zeroth order term: Notice we can write (P3) as $H(x, u, Du) = f(x)$ where

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Zeroth order term: Notice we can write (P3) as $H(x, u, Du) = f(x)$ where

$$H(x, z, p) = \prod_{i=1}^{n}(z + nx_i p_i)$$

Therefore

$$\begin{aligned}
\frac{\partial}{\partial z} H(x, z, p) &= H(x, z, p) \sum_{i=1}^{n} \frac{1}{z + nx_i p_i} \\
&\geq nH(x, z, p) \left(\prod_{i=1}^{n} \frac{1}{z + nx_i p_i} \right)^{\frac{1}{n}} \\
&= nH(x, z, p)^{\frac{n-1}{n}} \\
&= nf(x)^{\frac{n-1}{n}}.
\end{aligned}$$

Proof requires $f \geq m > 0$.

Convergence rate for (S2)

$$\text{(P3)} \quad \prod_{i=1}^{n}(w + nx_i w_{x_i}) = f \quad \text{in } (0,1]^n$$

$$\text{(P2)} \begin{cases} v_{x_1} \cdots v_{x_n} = v^{n-1} f & \text{in } (0,1]^n \\ v = 0 & \text{on } \Gamma \end{cases}$$

Recall $v = x_1 \cdots x_n w^n$. This identity approximately holds for w_h and v_h:

Lemma

Assume $f \in C^{0,1}([0,1]^n)$ and $f > 0$. Then

$$|x_1 \cdots x_n w_h(x)^n - v_h(x)| \leq Ch \quad \textit{for all } x \in [0,1]_h^n, \tag{8}$$

where

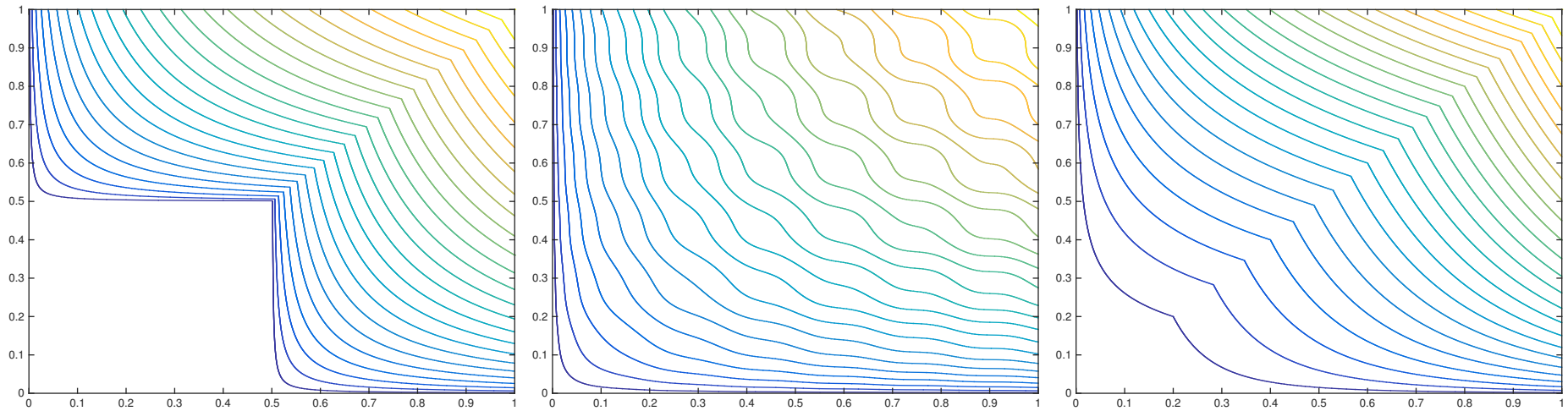
$$C = C\Big(n, [f]_{1;[0,1]^n}, \min_{[0,1]^n} f\Big).$$

The rate of convergence $|n^n v_h(x) - u(x)^n| \leq C\sqrt{h}$ follows from the lemma and the rate

$$|w_h(x) - w(x)| \leq C\sqrt{h}.$$

Outline

Test cases



(a) u_1 (b) u_2 (c) u_3

We consider 3 test cases:

$$u_1(x) = n \max_{i \in \{1,\dots,n\}} \left\{ \left(x_i - \frac{1}{2} \right)_+ \prod_{j \neq i} x_j \right\}^{\frac{1}{n}}$$

$$u_2(x) = \frac{1}{k+1} (x_1 \cdots x_n)^{\frac{1}{n}} \left(\sum_{j=1}^{n} \sin(k x_j)^2 + nk \right) \qquad (k = 20)$$

$$u_3(x) = n (x_1 \cdots x_n)^{\frac{1}{n}} \left(C \max\{x_1, \dots, x_n\} + \sum_{j=1}^{n} x_j \right) \qquad (C = 10)$$

n=2	(S1)		(S2)		(S3)	
Mesh size h	ℓ^∞ Error	Order	ℓ^∞ Error	Order	ℓ^∞ Error	Order
2.5×10^{-2}	7.1×10^{-2}		2.1×10^{-2}		6.7×10^{-2}	
6.3×10^{-3}	3.4×10^{-2}	0.54	5.7×10^{-3}	0.93	3.3×10^{-2}	0.51
1.6×10^{-3}	1.6×10^{-2}	0.51	1.5×10^{-3}	0.97	1.6×10^{-2}	0.50
3.9×10^{-4}	8.2×10^{-3}	0.50	3.8×10^{-4}	0.98	8.2×10^{-3}	0.50
9.8×10^{-5}	4.1×10^{-3}	0.50	9.7×10^{-5}	0.99	4.1×10^{-3}	0.50
2.4×10^{-5}	2.0×10^{-3}	0.50	2.4×10^{-5}	1.00	2.0×10^{-3}	0.50

n=3	(S1)		(S2)		(S3)	
Mesh size h	ℓ^∞ Error	Order	ℓ^∞ Error	Order	ℓ^∞ Error	Order
5×10^{-2}	3.1×10^{-1}		9.1×10^{-2}		2.1×10^{-1}	
2.5×10^{-2}	2.3×10^{-1}	0.41	5.3×10^{-2}	0.79	1.7×10^{-1}	0.34
1.3×10^{-2}	1.8×10^{-1}	0.38	3.0×10^{-2}	0.80	1.3×10^{-1}	0.34
6.3×10^{-3}	1.4×10^{-1}	0.36	1.7×10^{-2}	0.82	1.1×10^{-1}	0.33
3.1×10^{-3}	1.1×10^{-1}	0.35	9.5×10^{-3}	0.84	8.5×10^{-2}	0.33
1.6×10^{-3}	8.5×10^{-2}	0.34	5.3×10^{-3}	0.85	6.7×10^{-2}	0.33

n=4	(S1)		(S2)		(S3)	
Mesh size h	ℓ^∞ Error	Order	ℓ^∞ Error	Order	ℓ^∞ Error	Order
2.5×10^{-1}	1.1×10^{-0}		3.8×10^{-1}		4.9×10^{-1}	
1.3×10^{-1}	7.9×10^{-1}	0.46	2.4×10^{-1}	0.68	4.1×10^{-1}	0.26
6.3×10^{-2}	6.0×10^{-1}	0.40	1.5×10^{-1}	0.64	3.5×10^{-1}	0.25
3.1×10^{-2}	4.7×10^{-1}	0.36	9.5×10^{-2}	0.70	2.9×10^{-1}	0.25
1.6×10^{-2}	3.7×10^{-1}	0.32	5.7×10^{-2}	0.72	2.5×10^{-1}	0.25
7.8×10^{-3}	3.1×10^{-1}	0.29	3.4×10^{-2}	0.75	2.1×10^{-1}	0.25

n=2	(S1)		(S2)		(S3)	
Mesh size h	ℓ^∞ Error	Order	ℓ^∞ Error	Order	ℓ^∞ Error	Order
2.5×10^{-2}	9.5×10^{-2}		2.4×10^{-2}		2.4×10^{-2}	
6.3×10^{-3}	4.6×10^{-2}	0.53	6.1×10^{-3}	0.99	5.9×10^{-3}	1.01
1.6×10^{-3}	2.3×10^{-2}	0.50	1.6×10^{-3}	0.97	1.4×10^{-3}	1.02
3.9×10^{-4}	1.1×10^{-2}	0.50	4.1×10^{-4}	0.98	3.5×10^{-4}	1.02
9.8×10^{-5}	5.6×10^{-3}	0.50	1.0×10^{-4}	0.99	8.8×10^{-5}	1.00
2.4×10^{-5}	2.8×10^{-3}	0.50	2.6×10^{-5}	1.00	2.2×10^{-5}	0.99

n=3	(S1)		(S2)		(S3)	
Mesh size h	ℓ^∞ Error	Order	ℓ^∞ Error	Order	ℓ^∞ Error	Order
5×10^{-2}	3.6×10^{-1}		6.6×10^{-2}		5.6×10^{-2}	
2.5×10^{-2}	2.8×10^{-1}	0.39	4.8×10^{-2}	0.46	4.0×10^{-2}	0.48
1.3×10^{-2}	2.2×10^{-1}	0.36	2.4×10^{-2}	1.02	2.0×10^{-2}	1.01
6.3×10^{-3}	1.7×10^{-1}	0.35	1.2×10^{-2}	0.94	1.0×10^{-2}	0.96
3.1×10^{-3}	1.3×10^{-1}	0.34	6.2×10^{-3}	0.98	5.3×10^{-3}	0.97
1.6×10^{-3}	1.1×10^{-1}	0.34	3.2×10^{-3}	0.96	2.7×10^{-3}	0.96

n=4	(S1)		(S2)		(S3)	
Mesh size h	ℓ^∞ Error	Order	ℓ^∞ Error	Order	ℓ^∞ Error	Order
2.5×10^{-1}	1.6×10^{-0}		4.0×10^{-1}		3.1×10^{-1}	
1.3×10^{-1}	1.2×10^{-0}	0.42	3.9×10^{-1}		3.8×10^{-1}	
6.3×10^{-2}	6.9×10^{-1}	0.79	1.4×10^{-1}	1.49	1.1×10^{-1}	1.81
3.1×10^{-2}	5.3×10^{-1}	0.37	7.2×10^{-2}	0.93	5.8×10^{-2}	0.88
1.6×10^{-2}	4.3×10^{-1}	0.30	3.7×10^{-2}	0.94	3.0×10^{-2}	0.96
7.8×10^{-3}	3.4×10^{-1}	0.28	1.9×10^{-2}	0.98	1.5×10^{-2}	0.96

n=2	(S1)		(S2)		(S3)	
Mesh size h	ℓ^∞ Error	Order	ℓ^∞ Error	Order	ℓ^∞ Error	Order
2.5×10^{-2}	8.3×10^{-2}		7.5×10^{-2}		3.1×10^{-2}	
6.3×10^{-3}	4.2×10^{-2}	0.49	1.9×10^{-2}	1.00	8.0×10^{-3}	0.98
1.6×10^{-3}	2.1×10^{-2}	0.50	4.7×10^{-3}	1.00	2.0×10^{-3}	1.00
3.9×10^{-4}	1.1×10^{-2}	0.50	1.2×10^{-3}	1.00	5.0×10^{-4}	1.00
9.8×10^{-5}	5.3×10^{-3}	0.50	2.9×10^{-4}	1.00	1.3×10^{-4}	1.00
2.4×10^{-5}	2.7×10^{-3}	0.50	7.4×10^{-5}	1.00	3.1×10^{-5}	1.00

n=3	(S1)		(S2)		(S3)	
Mesh size h	ℓ^∞ Error	Order	ℓ^∞ Error	Order	ℓ^∞ Error	Order
5×10^{-2}	3.0×10^{-1}		2.6×10^{-1}		1.3×10^{-1}	
2.5×10^{-2}	2.5×10^{-1}	0.31	1.3×10^{-1}	0.96	6.8×10^{-2}	0.95
1.3×10^{-2}	2.0×10^{-1}	0.32	6.7×10^{-2}	0.99	3.5×10^{-2}	0.97
6.3×10^{-3}	1.6×10^{-1}	0.33	3.3×10^{-2}	0.99	1.8×10^{-2}	0.98
3.1×10^{-3}	1.2×10^{-1}	0.33	1.7×10^{-2}	1.00	8.8×10^{-3}	0.99
1.6×10^{-3}	9.9×10^{-2}	0.33	8.4×10^{-3}	1.00	4.4×10^{-3}	1.00

n=4	(S1)		(S2)		(S3)	
Mesh size h	ℓ^∞ Error	Order	ℓ^∞ Error	Order	ℓ^∞ Error	Order
2.5×10^{-1}	8.5×10^{-1}		1.4×10^{-0}		6.3×10^{-1}	
1.3×10^{-1}	6.6×10^{-1}	0.37	7.7×10^{-1}	0.84	3.8×10^{-1}	0.73
6.3×10^{-2}	5.5×10^{-1}	0.26	4.1×10^{-1}	0.89	2.1×10^{-1}	0.86
3.1×10^{-2}	4.6×10^{-1}	0.25	2.2×10^{-1}	0.94	1.1×10^{-1}	0.91
1.6×10^{-2}	3.9×10^{-1}	0.25	1.1×10^{-1}	0.98	5.6×10^{-2}	0.96
7.8×10^{-3}	3.2×10^{-1}	0.25	5.5×10^{-2}	0.99	2.8×10^{-2}	0.99

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