Mathematics of Image and Data Analysis
Math 5467

Lecture 16: The Sampling Theorem and Cosine Transform

Instructor: Jeff Calder
Email: jcalder@umn.edu

http://www-users.math.umn.edu/~jwcalder/5467S21

Last time

- Mult-dimensional DFT
- Image denoising

Today

- The sampling theorem
- Discrete Cosine Transform

The Sampling Theorem

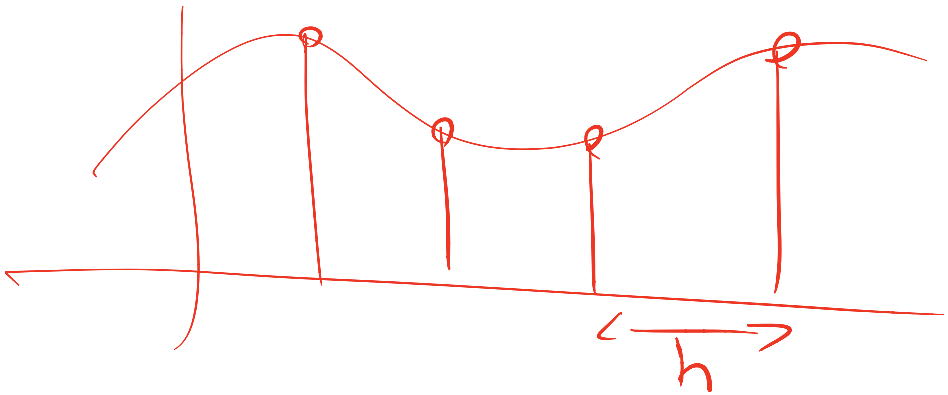
If a signal $u : \mathbb{R} \to \mathbb{R}$ contains no frequencies greater than σ_{max}, then u can be perfectly reconstructed from evenly spaced samples provided the sampling frequency is greater than the Nyquist rate $2\sigma_{max}$ and we have the Sinc Interpolation formula

$$u(t) = \sum_{j=-\infty}^{\infty} u(jh) \operatorname{sinc}\left(\frac{t - jh}{h}\right), \tag{1}$$

where h is the sampling period and

$$\frac{1}{h} > 2\sigma_{max}$$

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$



$$\int_{-\infty}^{\infty} u(t)\, e^{2\pi i t k}\, dt = 0$$

if $|k| \geq \sigma_{max}$

The Sampling Theorem

If a signal $u : \mathbb{R} \to \mathbb{R}$ contains no frequencies greater than σ_{max}, then u can be perfectly reconstructed from evenly spaced samples provided the sampling frequency is greater than the Nyquist rate $2\sigma_{max}$ and we have the Sinc Interpolation formula

$$u(t) = \sum_{j=-\infty}^{\infty} u(jh) \operatorname{sinc}\left(\frac{t - jh}{h}\right), \tag{2}$$

where h is the sampling period and

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

- The sampling frequency is $\frac{1}{h}$, so the Nyquist rate condition for the Sampling Theorem is that $\frac{1}{h} > 2\sigma_{max}$, or $h < \frac{1}{2\sigma_{max}}$.
- At sampling intervals $h > \frac{1}{2\sigma_{max}}$, high frequencies are aliased to lower frequencies, creating distortion.
- CD quality audio samples at a rate of 44.1 kHz, which was chosen to capture frequencies up to 22.05 kHz, higher than most humans can hear.

The Sampling Theorem (periodic version)

Let $u : \mathbb{R} \to \mathbb{R}$ be periodic with period 1, and assume u has no frequency larger than σ_{max}, where σ_{max} is a positive integer. This means that the signal u has the Fourier Series representation

$c_k \in \mathbb{C}$

$$u(t) = \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k \, e^{2\pi i k t}. \tag{3}$$

Fourier series

Theorem 1. *Suppose that u is given by (3) and let $h = 1/n$ for $n \in \mathbb{N}$ with $n > 2\sigma_{max}$. Assume also that n is odd. Then $u(t)$ can be reconstruted from its evenly spaced samples $u(jh)$ and furthermore we have*

$$u(t) = \sum_{j=0}^{n-1} u(jh) S\left(\frac{t - jh}{h}\right), \tag{4}$$

where $S(t)$ is given by

$$S(t) = \frac{\mathrm{sinc}(t)}{\mathrm{sinc}(ht)}.$$

Sinc kernel

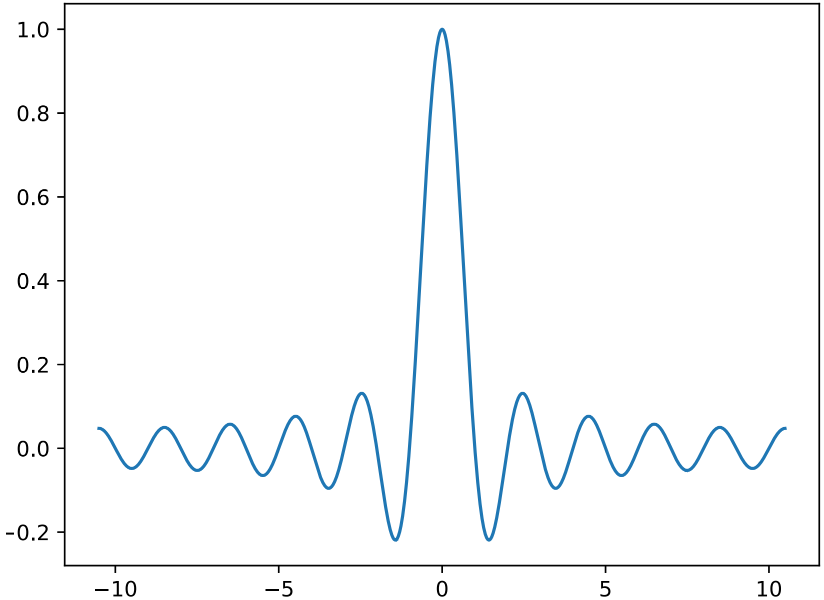


Figure 1: Depiction of the Sinc-like kernel $S(t) = \operatorname{sinc}(t)/\operatorname{sinc}(ht)$ for $n = 21$ and $h = 1/21$. The kernel is periodic with period $n = 21$.

Proof of Sampling Theorem

$$u(t) = \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k e^{2\pi i k t} \quad , \quad h = \frac{1}{n}$$

Sample u: $\quad f(j) = u(jh) = u\left(\frac{j}{n}\right)$

$f : \mathbb{Z} \to \mathbb{R}$ is n-periodic

So $\quad f \in L^2(\mathbb{Z}_n)$

Q: Can we recover c_k from f?

$$f(j) = u\left(\frac{j}{n}\right) = \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k e^{2\pi i k j/n}$$

$$= \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k \omega^{kj}, \quad \omega = e^{2\pi i/n}$$

Try taking inner product : $u_\ell(k) = \omega^{k\ell}$

$$\langle f, u_\ell \rangle = \sum_{j=0}^{n-1} f(j)\,\omega^{-j\ell}, \quad \overline{u_\ell}(k) = \omega^{-k\ell}$$

$-\frac{n}{2} \le \ell \le \frac{n}{2}$

$$= \sum_{j=0}^{n-1} \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k \omega^{j(k-\ell)}$$

$w = e^{2\pi i/n}$

$w^n = 1$

$$(\phi) = \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k \underbrace{\sum_{j=0}^{n-1} w^{j(k-\ell)}}_{\cancel{\overset{?}{=}} \begin{cases} 0, & k \neq \ell \\ n, & k = \ell \end{cases}} \cancel{\overset{?}{=}} n c_\ell$$

Recall $w^n = 1$

If $k - \ell = pn$

then $w^{j(k-\ell)} = w^{jpn}$

$= (w^n)^{jp} = 1.$

$$= n \sum_{k=-\sigma_{max}}^{\sigma_{max}} c_k \, \delta_{\{k = \ell \bmod n\}}$$

$k \equiv l \mod n$ means $k - l = pn$, $p \in \mathbb{Z}$.

If $\boxed{2\sigma_{max} < n}$ $\Leftrightarrow$ $2\sigma_{max} \le n-1$

then the sum above has only one term c_l. In this case

$$\langle f, u_l \rangle = n c_l, \quad c_l = \frac{1}{n}\langle f, u_l \rangle$$

and

$$u(t) = \frac{1}{n} \sum_{k=-\sigma_{max}}^{\sigma_{max}} \langle f, u_k \rangle e^{2\pi i k t}$$

$$= \frac{1}{n} \sum_{k=-\left(\frac{n-1}{2}\right)}^{\frac{n-1}{2}} \langle f, u_k \rangle e^{2\pi i k t} \quad , \sigma_{max} \le \frac{n-1}{2}$$

since $\langle f, u_k \rangle = 0$ for

$$\sigma_{max} \le |k| \le \frac{n-1}{2}.$$

$$= \frac{1}{n} \sum_{k=-\left(\frac{n-1}{2}\right)}^{\frac{n-1}{2}} \sum_{j=0}^{n-1} f(j) e^{-2\pi i j k/n} e^{2\pi i k t}$$

$f(j) = u(jh)$

$$u(t) = \sum_{j=0}^{n-1} u(jh) \left(\frac{1}{n} \sum_{k=-\left(\frac{n-1}{2}\right)}^{\frac{n-1}{2}} e^{2\pi i k (t - jh)} \right)$$

$\frac{1}{n} = h$

$$= \frac{\operatorname{sinc}(n(t-jh))}{\operatorname{sinc}(t)}$$

$$u(t) = \sum_{j=0}^{n-1} u(jh) \underbrace{\frac{\operatorname{sinc}(n(t-jh))}{\operatorname{sinc}(t-jh)}}_{S\left(\frac{t-jh}{h}\right)}$$

$n = \frac{1}{h}$

$$S(t) = \frac{\operatorname{sinc}(t)}{\operatorname{sinc}(ht)}$$

∎

Discrete Cosine Transform

It is often useful in practical applications to avoid complex numbers and work with real-valued transformations. If $f \in L^2(\mathbb{Z}_n)$ is *real-valued* then the Fourier Inversion Theorem yields

$$\mathcal{D}f(\ell) = \sum_{j=0}^{n-1} f(j)\, e^{-2\pi i j\ell/n}$$

$$\begin{aligned}
f(k) &= \frac{1}{n}\sum_{\ell=0}^{n-1} \mathcal{D}f(\ell) e^{2\pi i k\ell/n} \\
&= \frac{1}{n}\sum_{\ell=0}^{n-1}\sum_{j=0}^{n-1} f(j) e^{-2\pi i j\ell/n} e^{2\pi i k\ell/n} \\
&= \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\sum_{j=0}^{n-1} f(j)\cos(2\pi j\ell/n)\right)\cos(2\pi k\ell/n) \\
&\qquad + \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\sum_{j=0}^{n-1} f(j)\sin(2\pi j\ell/n)\right)\sin(2\pi k\ell/n).
\end{aligned}$$

proof next slide

f real-valued.

$$e^{-2\pi i j\ell/n}\, e^{2\pi i k\ell/n}$$

$$x = -2\pi j\ell/n$$
$$y = 2\pi k\ell/n$$

$$= (\cos x + i \sin x)(\cos y + i \sin y)$$

$$= \boxed{\cos x \cos y - \sin x \sin y} \quad \text{keep}$$

$$+ \underbrace{i(\sin x \cos y + \cos x \sin y)}_{\text{vanishes when summed above, since } f(k)\in\mathbb{R}.}.$$

Even/odd extensions

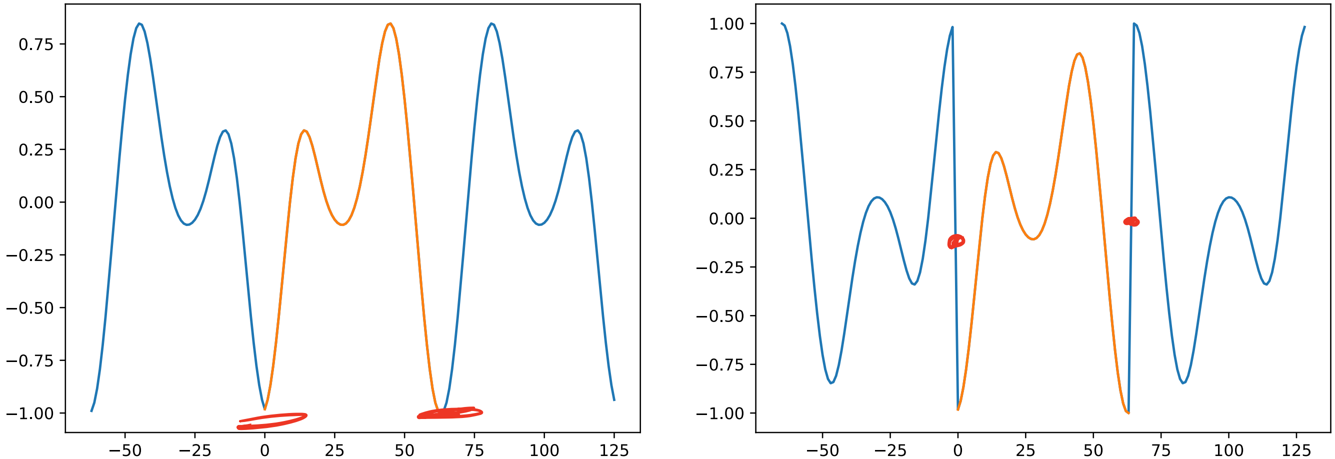
Let $f : \mathbb{Z}_n \to \mathbb{R}$. We define the even extension $f_e : \mathbb{Z}_{2(n-1)} \to \mathbb{R}$ by

$$f_e(k) = \begin{cases} f(k), & \text{if } 0 \le k \le n-1, \\ f(2(n-1)-k), & \text{if } n \le k \le 2(n-1)-1. \end{cases} \tag{5}$$

The odd extension $f_o : \mathbb{Z}_{2(n+1)} \to \mathbb{R}$ is defined by

$$f_o(k) = \begin{cases} 0, & \text{if } k = 0 \\ f(k-1), & \text{if } 1 \le k \le n, \\ 0, & \text{if } k = n+1 \\ -f(2(n+1)-1-k), & \text{if } n+2 \le k \le 2(n+1)-1. \end{cases} \tag{6}$$

Even/odd extensions



(a) Even extension

(b) Odd extension

Figure 2: Example of the even and odd extensions of a signal on $\mathbb{Z}_{64}$

Discete Cosine Transform

Recall

$$f(k) = \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\sum_{j=0}^{n-1} f(j)\cos(2\pi j\ell/n)\right)\cos(2\pi k\ell/n) + \frac{1}{n}\sum_{\ell=0}^{n-1}\left(\sum_{j=0}^{n-1} f(j)\sin(2\pi j\ell/n)\right)\sin(2\pi k\ell/n). \tag{*}$$

We now apply the representation formula above to the even extension f_e, taking $2(n-1)$ in place of n, to obtain the Discrete Cosine Transform

$$f(k) = \frac{1}{2(n-1)}(A_0 + (-1)^k A_{n-1}) + \frac{1}{n-1}\sum_{\ell=1}^{n-2} A_\ell \cos\left(\frac{\pi k\ell}{n-1}\right),$$

where

$$A_\ell = f(0) + (-1)^\ell f(n-1) + 2\sum_{k=1}^{n-2} f(k)\cos\left(\frac{\pi k\ell}{n-1}\right).$$

Apply (#) to f_e

$$f_e(k) = \frac{1}{2(n-1)} \sum_{l=0}^{2(n-1)-1} A_l \cos\left(\frac{\cancel{2}\pi k l}{\cancel{2}(n-1)}\right)$$

$$+ \frac{1}{2(n-1)} \sum_{l=0}^{2(n-1)-1} B_l \sin\left(\frac{\cancel{2}\pi k l}{\cancel{2}(n-1)}\right)$$

$$A_l = \sum_{j=0}^{2(n-1)-1} f_e(j) \cos\left(\frac{\cancel{2}\pi j l}{\cancel{2}(n-1)}\right)$$

$$B_l = \sum_{j=0}^{2(n-1)-1} f_e(j) \sin\left(\frac{\cancel{2}\pi j l}{\cancel{2}(n-1)}\right)$$

Claim: $B_\ell = 0$ (since f_e is even)

exercise!

Let's compute A_ℓ.

$$f_e(k) = \begin{cases} f(k), & 0 \le k \le n-1 \\ f(2(n-1)-k), & n \le k \le 2(n-1)-1 \end{cases}$$

$$A_\ell = \sum_{j=0}^{n-1} f(j)\cos\left(\frac{\pi j \ell}{n-1}\right) + \underbrace{\sum_{j=n}^{2(n-1)-1} f(2(n-1)-j)\cos\left(\frac{\pi j \ell}{n-1}\right)}_{C}$$

$$k = 2(n-1) - j \qquad j = n \iff k = n-2$$

$$j = 2(n-1) - k \qquad j = 2(n-1) - 1 \iff k = 1$$

$$C = \sum_{k=1}^{n-2} f(k) \cos\left(\pi \frac{(2(n-1) - k)\ell}{n-1}\right)$$

$$= \cos\left(2\pi\ell - \frac{\pi k \ell}{n-1}\right)$$

$$= \cos\left(-\frac{\pi k \ell}{n-1}\right) = \cos\left(\frac{\pi k \ell}{n-1}\right)$$

$$C = \sum_{k=1}^{n-2} f(k) \cos\left(\frac{\pi k \ell}{n-1}\right)$$

$$A_\ell = f(0) + f(n-1)\overbrace{\cos(\pi \ell)}^{=(-1)^\ell}$$

$$+ 2\sum_{j=1}^{n-2} f(j)\cos\left(\frac{\pi j \ell}{n-1}\right)$$

Rest left as exercise

(see class notes).

Discrete Sine Transform

Using the odd extension we get the Discrete Sine Transform

$$f(k) = \frac{1}{n+1} \sum_{\ell=0}^{n-1} B_\ell \sin\left(\frac{\pi(k+1)(\ell+1)}{n+1}\right), \tag{7}$$

where

$$B_\ell = 2 \sum_{k=0}^{n-1} f(k) \sin\left(\frac{\pi(k+1)(\ell+1)}{n+1}\right),$$

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