**Image Diffusion and Sharpening Via High-Order Sobolev Gradient Flows**

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**Motivation**

Many common image processing PDE can be equivalently interpreted as the $L^2$ gradient descent equations on appropriately chosen energy functionals.

**Heat Equation**

$$\frac{\partial u}{\partial t} = \Delta u \iff u \mapsto \int \|\nabla u\|^2$$  \hspace{1cm} (1)

**Perona-Malik Equation**

$$\frac{\partial u}{\partial t} = \text{div}(g(\|\nabla u\|^2)\nabla u) \iff u \mapsto \int g(\|\nabla u\|^2)$$  \hspace{1cm} (2)

**Youx-Kaveh Equation**

$$\frac{\partial u}{\partial t} = \Delta g(\|\nabla u\|^2) \iff u \mapsto \int g(\|\nabla u\|^2)$$  \hspace{1cm} (3)

Motivated by recent work in active contour applications we ask the following questions:

- How do these PDE change when we modify the metric from $L^2$ to a Sobolev metric?
- Are the resulting PDE well-posed and what properties do they have?
- How can we numerically compute solutions?
- Do these PDE have any useful properties for image processing?

This work answers these questions for the heat equation.

**Sobolev Gradient Descent**

We define the following inner product on $H^k_0(\Omega)$

$$\langle u, v \rangle_{H^k_0(\Omega)} = \sum_{|\alpha| \leq k} \int \nabla^\alpha u \cdot \nabla^\alpha v$$  \hspace{1cm} (4)

where the positive integers $c_\alpha \geq 1$ are chosen so that upon integrating by parts we have

$$\langle u, v \rangle_{H^k_0(\Omega)} = (\|\Delta^k u\|, v)$$  \hspace{1cm} (5)

The norm induced by this inner product is clearly equivalent to the standard $L^2$ norm. We now recall how the heat equation can be interpreted as the $L^2$ gradient descent equation on $H^k_0$.

$$\frac{d}{dt} E[u](t) = \int \nabla u \cdot \nabla v$$  \hspace{1cm} (6)

where $E[u](t) = \int \|\nabla u\|^2$.

**Sobolev Kernels $S_{\lambda}$**

In order to compute a Sobolev gradient, we need to solve the partial differential equation

$$(\|\Delta^{k} u\|, v) = f,$$  \hspace{1cm} (11)

where $f \in L^2(\mathbb{R}^n)$, $\lambda > 0$ and $k \geq 1$. The Green’s function for (11) is the Bessel function

$$S_{\lambda}(x) = \frac{1}{2\pi} e^{-t\frac{\sigma^2}{2\lambda}}$$  \hspace{1cm} (12)

Hence, we can compute $u = (\|\Delta^{k} u\|, v) f$ via the convolution $u = S_{\lambda} * f$. If we set $\lambda = \frac{\sigma^2}{2\lambda}$ for some $\sigma > 0$ and take $k \to \infty$ we obtain

$$\lim_{k \to \infty} S_{\lambda}(x) = G_{\sigma}(x) = \frac{1}{2\lambda} e^{-x^2/2\lambda}.$$  \hspace{1cm} (13)

Hence as $k \to \infty$, the Sobolev gradients tend towards Gaussian smoothed $L^2$ gradients.

**Theorem (Existence and Uniqueness)**

Let $m \geq 0$. Then for every $u_0 \in H^0_0(\Omega)$, there exists a unique $u \in C^1([0, \infty[; H^m_0(\Omega))$ solving (10)

$$\frac{du}{dt} = c(\|\Delta^{k} u\|)^{-\cdot} \Delta u, \quad \text{in} \quad \Omega \times [0, \infty[,$$  \hspace{1cm} (10)

for $t = 0$ where $c \in \mathbb{R}$ is a constant which controls the direction of diffusion. For $m = 0$ we identify $H_0^k(\Omega)$ with $L^2(\Omega)$.

**Remark**

Since (10) is well-posed for $c < 0$ we can consider reversing the Sobolev diffusion equations for image sharpening and deblurring.

**Theorem (Higher Regularity)**

If $u_0 \in C^k_0(\Omega)$ and $u$ solves (10) then $u(t) \in C^k(\Omega)$ for all $t \in [0, T]$.

**Remark**

The same result holds if we replace $C^k(\Omega)$ by $L^2(\Omega)$ for any $1 \leq \beta \leq \infty$.

**Figure**

Various Sobolev reverse diffusions ($c = -1$, $\lambda = 1$) shown when $u \mapsto \int \|\nabla u\|^2$ is reduced to 25%.

**Well-Posed Reverse Diffusion Under Increasing Sobolev Orders**

(a) $H^d$ diffusion  (b) $H^d$ diffusion  (c) $H^d$ diffusion

(a) $H^d$ diffusion  (b) $H^d$ diffusion  (c) $H^d$ diffusion

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(a) $H^d$ diffusion  (b) $H^d$ diffusion  (c) $H^d$ diffusion

Figure: Various Sobolev reverse diffusions ($c = -1$, $\lambda = 1$) shown when $u \mapsto \int \|\nabla u\|^2$ has doubled.